METU MATHEMATICS DEPARTMENT
REAL ANALYSIS
FEBRUARY 2014 - TMS EXAM

1.

a) State and prove Fatou's Lemma.

b) Show that Fatou's Lemma may not be true, even in the presence of uniform convergence.

(Hint: You may find $f_n(x) = -\frac{1}{n}\chi_{[0,n]}$ on $\mathbb{R}$ useful).

2. Let E be a measurable set of finite measure; (f_n) be a sequence of measurable real valued fuction on E. Show that for given $\epsilon > 0$ and $\delta > 0$ $\exists$ measurable A in E with measure $m(A) < \delta$ and a natural number N such that $\forall x \notin A$ and all $n \geq N$, $|f_n(x) - f(x)| < \epsilon$.

3.

a) Let $(X, \wedge, \mu)$ be a finite measure space. Let (f_n), (g_n) be two sequences of measurable functions and $f_n \to f$ in measure μ and $g_n \to g$ in measure μ. Show that $f_n g_n \to fg$ in measure.

b) By considering $f_n(X) = \sqrt{x^4 + \frac{x}{n}}$ and $f(x) = x^2$ on $(0, \infty)$ with Lebesgue measure, show that the conclusion may fail if the space has no finite measure.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Show that $\lim_{t\to\infty} \int f(x)\cos(xt)d\lambda(x) = 0$ when λ is the Lebesgue measure.