Preliminary Exam - February, 2017

Real Analysis

1) Let $(X, \mathcal{M}, \mu)$ be a measure space. Define $\mu^* : \mathcal{P}(X) \to [0, \infty]$ by $\mu^*(A) = \inf\{\mu(E) : E \in \mathcal{M}, A \subset E\}$. Prove that

a) μ^* is an outer measure on X.

b) $\forall A \in \mathcal{P}(X)\ \exists E \in \mathcal{M}$ such that $A \subset E$ and $\mu^*(A) = \mu(E)$.

2) Suppose that $\{f_n\}$ is a sequence of Lebesgue measurable functions on $[0, 1]$ such that $\lim_{n\to\infty} \int_0^1 |f_n| dm = 0$ and there is an integrable function g on $[0, 1]$ such that $|f_n|^2 \leq g$, for each n.

a) Prove that $\lim_{n\to\infty} \int_0^1 |f_n|^2 dm = 0$

b) Prove that if $\lim_n f_n = f$ exists a.e. then f integrable on $[0, 1]$ and $\int f dm = 0$

3) If f is a complex valued measurable function on $(X, \mathcal{M}, \mu)$, define

$$R_f = \{z : \mu(\{x : |f(x) - z| < \epsilon\}) > 0\ \forall \epsilon > 0\}$$

Show that

a) R_f is closed. b) If $f \in L^\infty$ then R_f is compact.

4) Let $(X, \mathcal{M}, \mu)$ be an arbitrary measure space and define ν on $\mathcal{M}$ by $\nu(A) = 0$ if $\mu(A) = 0$; and $\nu(A) = \infty$ if $\mu(A) > 0$.

a) Show that ν is a measure on X and $\nu << \mu$. b) Find $\frac{d\nu}{d\mu}$.