1. (15+10 pts) Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} 0 & \text{if } xy \in \mathbb{Q} \\ xy & \text{if } xy \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

(a) Is $f$ Riemann integrable on $[0, 1] \times [0, 1]$. If so, find its integral.
(b) Is $f$ Lebesgue integrable on $[0, 1] \times [0, 1]$. If so, find its integral.

2. (25pts) Let $m$ be the Lebesgue measure on $\mathbb{R}$, and $\mu(E) = \int_E e^{-x^2} dm(x)$.

(a) (15pts) Show that $m$ is absolutely continuous with respect to $\mu$.
(b) (10pts) Compute the Radon-Nikodym derivative $dm/d\mu$.

3. (25 points) Let $D = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, x < y < 1\}$ and

$$f(x, y) = y^{-3/2} \sin \left( \frac{\pi x}{2y} \right).$$

Is $f$ (Lebesgue) integrable on $D$? If so compute the double integral

$$\int \int_D f(x, y) dA$$

by referring all necessary Theorems.

4. (a) (10pts) State Fatou’s Lemma.
(b) (15pts) Let $f, f_n \in L^1(\mathbb{R})$, $f_n \to f$ pointwise on $\mathbb{R}$ and $\int |f_n| \to \int |f|$. Prove that $\int_E f_n \to \int_E f$ for any measurable set $E$. 