Real Analysis Preliminary Exam
Spring 2021

1. (15+10 pts) Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by
$$f(x,y) = \begin{cases} 0 & \text{if } \ xy \in \mathbb{Q} \\ xy & \text{if } xy \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$
   (a) Is f Riemann integrable on $[0,1] \times [0,1]$. If so, find its integral.
   (b) Is f Lebesgue integrable on $[0,1] \times [0,1]$. If so, find its integral.

2. (25pts) Let m be the Lebesgue measure on $\mathbb{R}$, and $\mu(E) = \int_E e^{-x^2} dm(x)$.
   (a) (15pts) Show that m is absolutely continuous with respect to μ.
   (b) (10pts) Compute the Radon-Nikodym derivative $dm/d\mu$.

3. (25 points) Let $D = \{(x,y) \in \mathbb{R}^2 : 0 < x < 1, x < y < 1\}$ and
$$f(x,y) = y^{-3/2} \sin\left(\frac{\pi x}{2y}\right).$$
   Is f (Lebesgue) integrable on D? If so compute the double integral
$$\int\int_D f(x,y) dA$$
   by referring all necessary Theorems.

4. (a) (10pts) State Fatou's Lemma.
   (b) (15pts) Let $f, f_n \in L^1(\mathbb{R})$, $f_n \to f$ pointwise on $\mathbb{R}$ and $\int |f_n| \to \int |f|$. Prove that $\int_E f_n \to \int_E f$ for any measurable set E.