1. (20 points) Let m be the Lebesgue measure on $\mathbb{R}$. Show that there exists an open dense set $\mathcal{U} \subseteq \mathbb{R}$ such that $m(\mathcal{U}) = 2022$.

2. (10+10+10 points) Let $\alpha : \mathbb{R} \to \mathbb{R}$ be defined as

$$\alpha(x) = \lfloor x \rfloor \text{ (greatest integer less than or equal to } x\text{).}$$

   (a) Show that α is increasing and right-continuous.

   (b) Write the definition and give a formula for $m^*(A)$ for any $A \subseteq \mathbb{R}$, where m^* is the outer measure associated to α.

   (c) What is the σ-algebra of m^*-measurable subsets?

3. (20 points) Let A be a Lebesgue measurable subset of $[0, 1] \times [0, 1]$ such that

$$m \times m(A) = 1,$$

   where m is the Lebesque measure on $\mathbb{R}$. Show that for almost every $x \in [0, 1]$, we have $m(s_x(A)) = 1$.
   Hint: Think about the complement of A in $[0, 1] \times [0, 1]$.

4. (30 points) Suppose $f, g : \mathbb{R}^n \to \mathbb{R}$ are continuous functions with compact support. Show that $f * g$ (convolution product of f and g) is also a continuous function with compact support.
   Hint: For continuity, show that for each sequence $\{x_m\}_{m=1}^{\infty}$ with $\lim\limits_{m\to\infty} x_m = x$ we have $\lim\limits_{m\to\infty} f * g(x_m) = f * g(x)$.