METU-MATHEMATICS DEPARTMENT
Graduate Preliminary Examinations

Topology

Duration: 3 hours

September 24, 2004

1. Let τ be a Hausdorff topology on X. Let $Cl_\tau(A)$ denote the closure of $A \subseteq X$ with respect to the topology τ

(**a**) Prove that the family

$$\mathfrak{B} = \{U - A \mid U \text{ is open, } Cl_\tau(A) \text{ is compact in } \tau\}$$

constitutes a basis for a new topology τ^* on X.

(**b**) Prove that τ^* is Hausdorff.

(**c**) Prove that $\tau = \tau^*$ iff all compact subsets of τ are finite.

2. If $A \times B$ is a compact subset of $X \times Y$ contained in an open set W in $X \times Y$, then there exists open sets $U \subseteq X$ and $V \subseteq Y$ such that $A \times B \subseteq U \times V \subseteq W$.

3. Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of Hausdorff infinite spaces, and $X = \prod_{i\in\mathbb{N}} X_i$ with the topology which has a base $\{\prod_{i\in\mathbb{N}} U_i \,:\, U_i \text{ open}\}$. Show that

(**a**) X is Hausdorff

(**b**) X is not seperable

(**c**) X has a closed discrete subspace which has the cardinality of $\mathbb{R}$.

4. Let $X = (\mathbb{R}^2, T)$ where T is the topology on $\mathbb{R}^2$ which is induced by the following metric

$$d((p,q),(r,s)) = \begin{cases} \mid q - s \mid & \text{if } p = r \\ \mid q \mid + \mid p - r \mid + \mid s \mid & \text{if } p \neq r \end{cases}$$

(**a**) Describe the relative topologies on $\mathbb{R} \times \{0\}$, $\{p\} \times \mathbb{R}$, $\mathbb{R} \times \{q\}$ for $q \neq 0$, and on an arbitrary line $ax + by = c$.

(**b**) Show that X is path connected.

(**c**) Find the number of path components of $X \setminus \{(p,q)\}$