METU - Mathematics Department
Graduate Preliminary Exam

Topology

Duration : 3 hours **Fall 2005**

1. Let (X, d_x), (Y, d_Y) be two metric spaces and let $f : X \to Y$ be continuous in the usual ϵ, δ definition.

   a) Prove that for every open set $V \subset Y$, the set $f^{-1}(V)$ is open in X.

   b) Suppose that X is compact and $y_0 \in Y - f(X)$. Prove that there is an open neighborhood V of $f(X)$ and a positive number r such that $V \cap B(y_0; r) = \emptyset$. Here, $B(y_0; r) = \{y \in Y : d_Y(y, y_0) < r\}$.

2. Let X be an infinite set with the f*inite complement topology* (ie. the collection of open sets is $\tau = \{A : X - A \text{ is finite, or } A = \emptyset\}$).

   a) Prove that every subset of X is compact.

   b) Prove that X is T_1 (ie. For every $x, y \in X$ with $x \neq y$, there are open sets U, V such that $x \in U - V$ and $y \in V - U$).

   Is X Hausdorff ? Is X metrizable ?

   c) If $X = \mathbb{R}$, find the closures and interiors of $(0, 1], [2, 3], \mathbb{Z}$.

3. **a)** Consider the following subsets of $\mathbb{R}^2$:

   $X = \{(\pm\frac{1}{n},\ y) : n \geq 1, 0 \leq y \leq 1\} \cup \{(x, 0) : |x| \leq 1\} \cup \{(0, y) : 0 \leq y \leq 1\}$,

   $Y = \{(\pm\frac{1}{n},\ y) : n \geq 1, 0 \leq y \leq 1\} \cup \{(x, 0) : |x| \leq 1\} \cup \{(-2, y) : 0 \leq y \leq 1\}$

   Show that in the topology induced from $\mathbb{R}^2$,

   (i) X is connected but not locally connected, and

   (ii) Y is locally connected.

b) Show that the image of a locally connected set under a continuous map is not necessarily locally connected.

Hint : Consider the map $f : Y \to X$, $f(a,b) = \begin{cases} (a,b) & \text{if } a \neq -2 \\ (0,b) & \text{if } a = -2. \end{cases}$

c) Show that a compact Hausdorff space is locally connected if and only if every open cover of it can be refined by a cover consisting of a finite number of connected spaces.

4. **a)** Show that a topological space X is regular if and only if for each $x \in X$ and any neighborhood U of x, there is a closed neighborhood V of x such that $V \subset U$.

   b) Let X be a regular space and let $\mathcal{D}$ be the family of all subsets of the form $\overline{\{x\}}$ where $\overline{\{x\}}$ denotes the closure of the point $x \in X$. Show that $\mathcal{D}$ is a partition of X.

   c) Show that, in the quotient topology induced by the projection

   $p : X \to \mathcal{D},\ p(x) = \overline{\{x\}}$, $\mathcal{D}$ is a regular Hausdorff space.