1. Let \((X, d_X), (Y, d_Y)\) be two metric spaces and let \(f : X \to Y\) be continuous in the usual \(\epsilon, \delta\) definition.

a) Prove that for every open set \(V \subset Y\), the set \(f^{-1}(V)\) is open in \(X\).

b) Suppose that \(X\) is compact and \(y_0 \in Y - f(X)\). Prove that there is an open neighborhood \(V\) of \(f(X)\) and a positive number \(r\) such that \(V \cap B(y_0; r) = \emptyset\). Here, \(B(y_0; r) = \{y \in Y : d_Y(y, y_0) < r\}\).

2. Let \(X\) be an infinite set with the finite complement topology (ie. the collection of open sets is \(\tau = \{A : X - A\) is finite, or \(A = \emptyset\}\)).

a) Prove that every subset of \(X\) is compact.

b) Prove that \(X\) is \(T_1\) (ie. For every \(x, y \in X\) with \(x \neq y\), there are open sets \(U, V\) such that \(x \in U - V\) and \(y \in V - U\).

Is \(X\) Hausdorff? Is \(X\) metrizable?

c) If \(X = \mathbb{R}\), find the closures and interiors of \((0, 1], [2, 3], \mathbb{Z}\).

3. a) Consider the following subsets of \(\mathbb{R}^2\):

\[X = \{(\pm \frac{1}{n}, y) : n \geq 1, 0 \leq y \leq 1\} \cup \{(x, 0) : |x| \leq 1\} \cup \{(0, y) : 0 \leq y \leq 1\},\]

\[Y = \{(\pm \frac{1}{n}, y) : n \geq 1, 0 \leq y \leq 1\} \cup \{(x, 0) : |x| \leq 1\} \cup \{(-2, y) : 0 \leq y \leq 1\}\]

Show that in the topology induced from \(\mathbb{R}^2\),

(i) \(X\) is connected but not locally connected, and

(ii) \(Y\) is locally connected.
b) Show that the image of a locally connected set under a continuous map is not necessarily locally connected.

Hint: Consider the map \( f : Y \to X, f(a, b) = \begin{cases} (a, b) & \text{if } a \neq -2 \\ (0, b) & \text{if } a = -2. \end{cases} \)

c) Show that a compact Hausdorff space is locally connected if and only if every open cover of it can be refined by a cover consisting of a finite number of connected spaces.

4. a) Show that a topological space \( X \) is regular if and only if for each \( x \in X \) and any neighborhood \( U \) of \( x \), there is a closed neighborhood \( V \) of \( x \) such that \( V \subset U \).

b) Let \( X \) be a regular space and let \( D \) be the family of all subsets of the form \( \{x\} \) where \( \{x\} \) denotes the closure of the point \( x \in X \). Show that \( D \) is a partition of \( X \).

c) Show that, in the quotient topology induced by the projection \( p : X \to D, p(x) = \{x\}, D \) is a regular Hausdorff space.