METU - Mathematics Department
Graduate Preliminary Exam

Topology

Duration : 3 hours **September 15, 2006**

1. Let X be an infinite set and $T = \{A \subset X : A = \emptyset \text{ or } X - A \text{ is finite}\}$. Let Y be a **subspace** of the topological space (X, T). Show that

   a) Y is T_2 if and only if Y is finite.
   b) Y is connected if and only if Y is infinite.
   c) Y is compact and separable.
   d) If the cardinality of Y is at least the cardinality of $\mathbb{R}$, then Y is path connected.

2. Let X, Y be topological spaces and $f : X \to Y$ be a continuous, open surjection and $\mathcal{B}$ be a base for the topological space X. Show that

   a) The set $\{f(U) : U \in \mathcal{B}\}$ is a base for the topological space Y.

   b) If X is locally connected, then Y is locally connected.

   c) If X is locally compact (not necessarily Hausdorff), then Y is locally compact.

3. Let $X \neq \emptyset$ be a topological space. We say $X_0 \subset X$ is very dense in X if the following correspondence between the sets of open subsets

$$\begin{gathered} \mathcal{O}(X) \to \mathcal{O}(X_0) \\ U \mapsto U \cap X_0 \end{gathered}$$

   is injective.

   a) (i) Show that a v.d. set is dense and give a **non-Hausdorff** example to show that the converse is not true.
   (ii) Determine the topology of X if $\{x\} \subset X$ is v.d.

   b) Show that if X is T_0 and contains a v.d. subset X_0 which is minimal (with respect to inclusion) in the set of nonempty closed subsets of X, then X consists of a single point.

   c) True or false ? Explain (prove the claim or give a counter example).

   If X is a connected topological space and if X_0 is v.d. in X, then X_0 is connected too.

4. Let $C(X,Y)$ be the set of all continuous functions between a locally compact topological space X and an arbitrary topological space Y. We define the **weak topology** on $C(X,Y)$ by taking as **subbase** the sets of the form

$$\{f \in C(X,Y) : f(K) \subset V\}$$

for compact subsets $K \subset X$ and open sets $V \subset Y$.

A) Suppose X is compact and Y is a metric space. Show that

a) the weak topology is defined by a metric d_W on $C(X,Y)$,

b) if Y is complete, then $(C(X,Y), d_W)$ is a complete metric space.

B) Show that if $g : X_1 \to X_2$ is a continuous map between locally compact spaces then the induced map

$$\begin{aligned} g^* : C(X_2, Y) &\to C(X_1, Y) \\ f &\mapsto f \circ g \end{aligned}$$

is continuous.