Topology

1. Let $X$ and $Y$ be two topological spaces and let $X \times Y$ be given the product topology.
   a) Suppose $K$ is a compact subset of $X$ and $A \subset X \times Y$ is an open set such that for some $y \in Y$, $K \times \{y\} \subset A$. Show that $y$ has a neighborhood $U \subset Y$ such that $K \times U \subset A$.
   b) (i) Suppose $X$ is compact. Prove that the projection $\pi : X \times Y \rightarrow Y$ is a closed map.
      (ii) Give an example to show that in (i) the compactness assumption is essential.

2. a) Let $X$ be a connected, Hausdorff and completely regular topological space. Prove that if $X$ has at least two points, then $X$ is uncountable.
   (Recall : A $T_1$-topological space $X$ is completely regular if for any given closed subset $Y$ of $X$ and for any $p \in X$, one can find a continuous function $f : X \rightarrow \mathbb{R}$ such that $f(p) = 0$ and $f|_Y = 1$.)
   b) Give an example of a Hausdorff completely regular space which is countable.

3. Prove that the one-point compactification of $\mathbb{R}^2$ (in its usual topology) is homeomorphic to the 2-sphere $S^2 \subset \mathbb{R}^3$ (the topology on $S^2$ is induced from $\mathbb{R}^3$).

4. True or false ? Prove the statement or give a counter example.
   a) Let $(X,d)$ be a metric space. Suppose that there exist a point $a \in X$ and a real number $\epsilon_0 > 0$ such that for all $\epsilon > \epsilon_0$ one has $B(a;\epsilon) = \overline{B}(a;\epsilon)$. Then $(X,d)$ is bounded.
   b) If $g : Y \rightarrow X$ is a continuous map into a discrete topological space, then $g$ is constant on each connected component of $Y$.
   c) Recall that a point $x \in X$ in a topological space $X$ is called a generic point if it is dense.
      (i) If $X$ has a unique generic point, then it is connected.
      (ii) A connected topological may have at most a unique generic point.