METU - Mathematics Department
Graduate Preliminary Exam-Fall 2007

Topology

1. Let X and Y be two topological spaces and let $X \times Y$ be given the product topology.

   a) Suppose K is a compact subset of X and $A \subset X \times Y$ is an open set such that for some $y \in Y$, $K \times \{y\} \subset A$. Show that y has a neighborhood $U \subset Y$ such that $K \times U \subset A$.

   b) (i) Suppose X is compact. Prove that the projection $\pi : X \times Y \to Y$ is a closed map.

   (ii) Give an example to show that in (i) the *compactness* assumption is essential.

2. a) Let X be a connected, Hausdorff and completely regular topological space. Prove that if X has at least two points, then X is uncountable.

   (Recall : A T_1-topological space X is *completely regular* if for any given closed subset Y of X and for any $p \in X$, one can find a continuous function $f : X \to \mathbb{R}$ such that $f(p) = 0$ and $f|_Y = 1$.)

   b) Give an example of a Hausdorff completely regular space which is countable.

3. Prove that the one-point compactification of $\mathbb{R}^2$ (in its usual topology) is homeomorphic to the 2-sphere $S^2 \subset \mathbb{R}^3$ (the topology on S^2 is induced from $\mathbb{R}^3$).

4. True or false ? Prove the statement or give a counter example.

   a) Let (X, d) be a metric space. Suppose that there exist a point $a \in X$ and a real number $\epsilon_0 > 0$ such that for all $\epsilon > \epsilon_0$ one has $B(a; \epsilon) = \overline{B(a; \epsilon)}$. Then (X, d) is bounded.

   b) If $g : Y \to X$ is a continuous map into a discrete topological space, then g is constant on each connected component of Y.

   c) Recall that a point $x \in X$ in a topological space X is called a *generic point* if it is dense.
   (i) If X has a unique generic point, then it is connected.
   (ii) A connected topological may have at most a unique generic point.