METU - Department of Mathematics
Graduate Preliminary Exam
General Topology

Fall 2008

1.

(A) In a topological space, is the intersection of two dense subsets always dense ?

(B) Let X be a topological space. Prove that the intersection of two open and dense subsets of X is open and dense.

(C) Let $\mathfrak{H}$ be the family subsets of X consisting of the open and dense subsets of X and the empty set. Prove that $\mathfrak{H}$ is a topology on X.

(D) Let $\widetilde{X}$ be the topological space with carrier set X and topology $\mathfrak{H}$. Let Y be a topological space and $\widetilde{Y}$ be defined similarly. If $f : X \longrightarrow Y$ is a continuous and open map, prove that $f : \widetilde{X} \longrightarrow \widetilde{Y}$ is continuous.

2.

(A) Prove that a closed subset of a compact topological space is compact.

(B) Prove that a compact subset of a Hausdorff topological space is closed.

(C) $\mathfrak{H}, \mathfrak{H}'$ be topologies on X such that $\mathfrak{H}' \subseteq \mathfrak{H}$. If $K \subseteq X$ is compact with respect to $\mathfrak{H}$, prove that K is compact with respect to $\mathfrak{H}'$ as well.

(D) Suppose that $\mathfrak{T}$ is a topology on X such that the topological space $(X, \mathfrak{T})$ is compact. If $\mathfrak{S}$ is a topology on X which is strictly coarser than $\mathfrak{T}$, (that is $\mathfrak{S} \subseteq \mathfrak{T}$ yet $\mathfrak{S} \neq \mathfrak{T}$) prove that the topological space $(X, \mathfrak{S})$ is not Hausdorff. (*Hint :* There exists $C \subseteq X$ which is closed with respect to $\mathfrak{T}$ but not closed with respect to $\mathfrak{S}$.)

3. Let $f : X \to Y$ be a continuous closed surjection. Prove the following statements:

i) If X is T_1 then Y is T_1.

ii) If X is normal then Y is normal.

iii) If Y is connected and $f^{-1}(y)$ is connected for each $y \in Y$ then X is connected.

iv) If $f^{-1}(y)$ is Lindelöf for each $y \in Y$ and Y is Lindelöf then X is Lindelöf,

4.Let $f : X \to Y$ be a closed open surjection.

a) Let $\varphi : X \to (0, 1)$ be a continuous function and $\Theta : Y \to \mathbb{R}$ be a function such that $\Theta(y) = \sup[\varphi(x) | f(x) = y]$ for each $y \in Y$. Show that θ is continuous.

b) Show that Y is regular (T_3) when X is regular and f is continuous.

c) Show that Y is $T_{3\frac{1}{2}}$ when X is $T_{3\frac{1}{2}}$ and f is continuous.