Preliminary Exam -September 2025

Topology

(30 pts) 1) Let X be a topological space and $A, B \subseteq X$, A is called regular open if $\overline{(A)}^0 = A$. B is called nowhere dense if $\overline{(B)}^0 = \emptyset$

i) Show that: If U, V are regular open then $U \cap V$ is regular open.

ii) Show that: If C,D are nowhere dense then $C \cup D$ are nowhere dense.

iii) Show that: If O is open then $\overline{O \cap K} = \overline{O \cap \overline{K}}$ for each $K \subseteq X$.

(15 pts) 2) Let X be a Hausdorff space such that each function $f : X \to X$ is continuous. Show that X is discrete space.

(15 pts) 3) Let $f : X \to Y$ be continuous surjection. Consider the following statement:

"If $B \subseteq Y, f^{-1}(B)$ is closed then B is closed". Show that

a) The satatement is true when X is compact Y is Hausdorff.

b) The statement may not be true when X is not compact.

c) The statement may not be true when Y is not Hausdorff.

(20 pts) 4) Let X be topological space $p \in X$. Define $C(p) = U\{C| \quad p \in C$, C is connected$\}$, $Q(p) = \bigcap\{0| \quad p \in O,$ O is open and closed$\}$

a) Show that: $C(p) \subseteq Q(p)$

b) Show that: $C(p) = Q(p)$ when X is compact and Hausdorff.

(20 pts) 5) Let I be an index set and $X = \prod_{i \in I}(0, 1)$ be the product space (with the product topology). Define a partial order on X by: $f \leq g \Leftrightarrow f(i) \leq g(i)$ for every $i \in$ I.

a) Show that $K_f = \{g \in X | g \leq f\}$ is closed subset of X for each $f \in X$.

b) If C compact subset of X, show that C has a minimal element with respect to $\leq$, i.e., there exists $f \in C$ such that if $g \in C$ and $g \leq f$, then $g = f$.