Graduate Preliminary Examination
Topology

Duration: 3 hours

1. Consider a topological space $X, Y$ and a continuous map $f : X \to Y$.
   a) Prove that $f^{-1}(B) \subseteq f^{-1}(B)$ for any subset $B$ of $Y$.
   b) Suppose that $f$ is also closed and surjective. Prove that $B = f(f^{-1}(B))$
      for any subset $B$ of $Y$.
   c) Suppose that $X$ is metrizable and $f$ is a closed, surjective and continuous map. Prove that for any subset $B$ of $Y$ and any $y \in B$
      there exists a sequence $y_n \in B$ such that $\lim y_n = y$.

2. a) Is the intersection of two dense subsets in a topological space always dense?
   b) Let $X$ be a topological space. Prove that the intersection of two open dense subsets of $X$ is open and dense.
   c) If $\mathcal{H}$ is the family of open dense subsets in $X$, prove that $\mathcal{H} = \mathcal{H} \cup \{a\}$
      is a topology on $X$.
   d) Let $X$ be the topological space which consists of the set $X$ with the topology $\mathcal{H}$ on $X$. Prove that a function $f : X \to \mathbb{R}$ is continuous if it is
      continuous.

3. Let $f$ be a continuous mapping of the compact space $X$ onto the
    Hausdorff space $Y$. Show that any mapping $g$ of $Y$ into $X$ for which $g \circ f$ is
    continuous must itself be continuous.

4. Consider the cylinder $S^1 \times I$ where $S^1$ the unit circle in $\mathbb{R}^2$ and
    $I = [0, 1]$. Identify $S^1 \times \{1\}$ to a point i.e. define an equivalence relation
~ on S^1 by letting (u,1) ~ (v,1) for all u,v ∈ S^1 and letting all other elements in S^1 × [0,1] be related only to itself. Show that the quotient space (S^1 × D^2/~), the so-called torus on S^1, is homeomorphic to the unit disc D^2 in R^2.