METU - Mathematics Department
Graduate Preliminary Exam

Topology

Duration : 3 hr. **Feb 9, 2007**

1. Let τ be a Hausdorff topology on a set $X \neq \emptyset$ and for $A \subset X$, let $\overline{A}$ denote the closure of A with respect to τ.

   a) Prove that the family

   $$\mathcal{B} = \{U - A : U \text{ is open}, \overline{A} \text{ is compact in } \tau\}$$

   constitutes a basis for a new topology $\tau*$.

   b) Prove that $\tau*$ is Hausdorff.

   c) Prove that $\tau = \tau*$ if and only all subsets of X which are compact in τ, are finite subsets.

2. Let X be an infinite set with the f*inite complement topology* (ie. the collection of open sets is $\tau = \{A : X - A \text{ is finite, or } A = \emptyset\}$).

   a) Prove that every subset of X is compact.

   b) Prove that X is T_1 (ie. For every $x, y \in X$ with $x \neq y$, there are open sets U, V such that $x \in U - V$ and $y \in V - U$).

   Is X Hausdorff ? Is X metrizable ?

   c) If $X = \mathbb{R}$, find the closures and interiors of $(0, 1], [2, 3], \mathbb{Z}$.

3. Let X be a Hausdorff topological space and let $X^* = X \cup \{\infty\}$, where ∞ is an ideal point not in X. Consider the following collection Ω^* of subsets of X^*

   (i) open sets in X

   (ii) sets of the form $X^* - S$ where S is a compact subset of X.

   Prove the following statements for the topological space (X^*, Ω^*) (**do not prove** that Ω^* defines a topology on X^*).

   a) (X^*, Ω^*) is compact.

   b) If X is locally compact, then X^* is Hausdorff.

   c) A continuous map $f : X \to Y$ between Hausdorff topological spaces extends to a map $f^* : X^* \to Y^*$, which is continuous if f is proper (that is, the inverse image under f of every compact subset of Y is compact).

   d) If Y is a locally compact Hausdorff space and if $f : X \to Y$ is proper, then f is a closed map.

4. Let (X, d) be a metric space. For a point x and for subspaces A, B in X, define $d(x, A) = \inf\{d(x, a) : a \in A\}$ and $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$.

   (a) Prove that the map $f : X \to \mathbb{R}$ defined by $f(x) = d(x, A)$ is continuous.

   (b) Prove that if A is compact, then there is a point $a_0 \in A$ such that $d(x, a_0) = d(x, A)$.

   (c) Prove that if A and B are compact, then there are points $a_0 \in A$ and $b_0 \in B$ such that $d(a_0, b_0) = d(A, B)$.

   (d) Prove that A and B are compact and disjoint, then there are disjoint open sets U and V in X such that U contains A and V contains B.

   (e) Show that the conclusion of (b) may not be true if A is not compact.