1. Consider the real line $\mathbb{R}$ with the usual topology. Let $\sim$ be the equivalence relation defined by
\[ x \sim y \text{ if and only if } x - y \in \mathbb{Q}. \]
Show that the quotient space $\mathbb{R}/\sim$ has uncountable number of elements and that its topology is trivial.

2. A **discrete valued map** on a topological space $X$ is a continuous map $X \to D$ into a discrete topological space $D$. **Show that**
   a) $X$ is connected if and only if every discrete valued map on $X$ is constant.
   b) the statement "$d(p) = d(q)$ for every discrete valued map $d$ on $X$" defines an equivalence relation on $X$ and that the corresponding equivalence classes are closed subsets of $X$.

3. Let $X$ and $Y$ be two topological spaces and let $X \times Y$ be given the product topology.
   a) Suppose $K$ is a compact subset of $X$ and $A \subset X \times Y$ is an open set such that for some $y \in Y$, $K \times \{y\} \subset A$. Show that $y$ has a neighborhood $U \subset Y$ such that $K \times U \subset A$.
   b) (i) Suppose $X$ is compact. Prove that the projection $\pi : X \times Y \to Y$ is a closed map.
      (ii) Give an example to show that in (i) the *compactness* assumption is essential.

4. Let $X$ be a Hausdorff topological space and let $X^* = X \cup \{\infty\}$, where $\infty$ is an ideal point not in $X$. Consider the following collection $\Omega^*$ of subsets of $X^*$
   (i) open sets in $X$
   (ii) sets of the form $X^* - S$ where $S$ is a compact subset of $X$. 
Prove the following statements for the topological space \((X^*, \Omega^*)\) (do not prove that \(\Omega^*\) defines a topology on \(X^*)\).

a) \((X^*, \Omega^*)\) is compact.

b) If \(X\) is locally compact, then \(X^*\) is Hausdorff.

c) A continuous map \(f : X \to Y\) between Hausdorff topological spaces extends to a map \(f^* : X^* \to Y^*\), which is continuous if \(f\) is proper (that is, the inverse image under \(f\) of every compact subset of \(Y\) is compact).

d) If \(X\) and \(Y\) are locally compact Hausdorff spaces and if \(f : X \to Y\) is proper and continuous, then \(f\) is a closed map.