GRADUATE PRELIMINARY EXAMINATION
GENERAL TOPOLOGY
Duration: 3 hours
February 10^{th}, 2009

1. A topological space X is said to be *hyperconnected* if every nonempty open set in X is dense. A topological space Y is said to be *ultraconnected* if $\overline{\{a\}} \cap \overline{\{b\}} \neq \emptyset$ for every $a, b \in Y$.

a) Prove that a hyperconnected topological space is connected.

b) Prove that an infinite set with the cofinite topology is hyperconnected but not ultraconnected.

c) Let Z be a topological space with more than two points and $p \in Z$ where $U \subseteq Z$ is open iff $U = Z$ or $p \notin U$. Prove that Z is ultraconnected but not hyperconnected.

d) Prove that an ultraconnected space Y is path connected by demonstrating that for any $a, b \in Y$ and any $p \in \overline{\{a\}} \cap \overline{\{b\}}$ the map $\lambda : [0,1] \to Y$ defined by

$$\lambda(t) = \begin{cases} a & \text{if } \ 0 \leq t < 1/2 \\ p & \text{if } \ t = 1/2 \\ b & \text{if } \ 1/2 < t \leq 1 \end{cases}$$

is continuous.

2. A continuous map $f : X \to Y$ is called a (topological) embedding if the map $\tilde{f} : X \to f(X)$ obtained by restricting the range of f is a homeomorphism (Here $f(X)$ has the subspace topology).

a) Show that $\mathbb{Q}$ cannot be embedded in $\mathbb{Z}$ (where both has the subspace topology of $\mathbb{R}$).

b) Let $\mathbb{R}_c$ denote the topological space whose underlying set is $\mathbb{R}$ and its open sets are complements of finite sets and the empty set.

(i) If $Y \subset \mathbb{R}_c$, describe open subsets of Y.

(ii) Show that $\mathbb{R}$ with its usual topology cannot be embedded in $\mathbb{R}_c$.

3. a) Let X and Y be metric spaces so that Y is compact. Show that the projection map $\pi_1 : X \times Y \to X$, where $\pi_1(x, y) = x$ for all $(x, y) \in X \times Y$, is a closed map.

b) Let $X = Y = \mathbb{R}$ equipped with the usual metric. Show that the projection map $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in Part (a) is <u>not</u> a closed map.

4. Let $A = \{A_i | i \in I\}$ be a cover for topological space X such that each $p \in X$ has a neighborhood N such that $\{i | N \cap A_i \neq \phi\}$ is finite and $f : X \to Y$ be a function where Y is a topological space.

a) Show that $\overline{\bigcup_{i \in I} B_i} = \bigcup_{i \in I} \overline{B_i}$ where $B_i \subseteq A_i$ for each $i \in I$.

b) Show that f is continuous when $f|A_i$ continuous and A_i is closed for each $i \in I$.

c) Show that $\{i | K \cap A_i \neq \phi\}$ is finite for any compact subset K of X.

d) Show that the component of a point $p \in X$ (maximal connected subset which contains p) is open when A_i is connected for each $i \in I$.