TOPOLOGY TMS EXAM
February 11 2015

Duration: 3 hours.

(1) Consider the set of real numbers $\mathbb{R}$. The collection of all open intervals (a,b) along with all sets of the form $(a,b) - K$ where $K = \{\frac{1}{n} | n \in \mathbb{Z}^+\}$ is a basis for a topology on $\mathbb{R}$ called the K-topology. Let $\mathbb{R}_K$ denote $\mathbb{R}$ with this topology.
(a) Compare $\mathbb{R}_K$ with the standard topology on $\mathbb{R}$.
(b) Is $\mathbb{R}_K$ compact? Is $[0,1]$ a compact subspace of $\mathbb{R}_K$?
(c) Is $\mathbb{R}_K$ Hausdorff?
(d) Is $\mathbb{R}_K$ regular?
(e) Is the quotient space of $\mathbb{R}_K$ obtained by collapsing the set K to a point Hausdorff? Is it T_1?

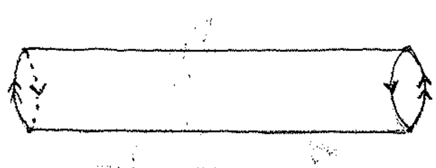
(2) Let X be a compact metric space and suppose that $f: X \to X$ is an isometry, that is $d(f(x), f(y)) = d(x,y)$ for all $x, y \in X$. Prove that f is a homeomorphism.

(3) Let $\mathbb{R}^\omega$ be the countably infinite product of $\mathbb{R}$ with itself and let $A \subset \mathbb{R}^\omega$ be defined by

$$A = \{(x_i) \in \mathbb{R}^\omega | \ x_i = 0 \text{ for all but finitely many } i\}.$$

(a) Prove that A is dense in $\mathbb{R}^\omega$ with the product topology.
(b) Prove that A is not dense in $\mathbb{R}^\omega$ with the box topology.

(4) Let $r: S^1 \to S^1$ be defined by $r(x,y) = (-x,y)$. The Klein bottle K is the quotient space of $[0,1] \times S^1$ under the following equivalence relation: $(0,(x,y)) \sim (1, r(x,y))$ for all $(x,y) \in S^1$ and $(t,(x,y))$ is not equal to anything except itself for $t \neq 0, 1$.



(a) Show that K is compact.
(b) Let $C_1 \subset K$ be (the image of) the circle $\{\frac{1}{3}\} \times S^1$, and let $C_2 \subset K$ be a small embedded circle inside $(\frac{1}{2}, \frac{3}{4}) \times S^1$. There is a continuous map $g: K \to \mathbb{R}^3$ as shown in the picture. The restriction of g to $K - C_1$ is injective so is the restriction to $K - C_2$, but $g(C_1) = g(C_2)$. Assuming that such a map g exists as described, use Urysohn's Lemma to construct a continuous map of K into $\mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$ which is an embedding. You may assume that K is Hausdorff.