Topology

TMS EXAM

February 16, 2017

Duration: 3 hr.

1. a) Show that every subspace of a second countable space is Lindelöf.

b) Show that any base for the open sets in a second countable space has a countable subfamily which is a base.

2. Show that continuous Hausdorff image of a compact locally connected space is compact and locally connected.

3. a) Define a (topological) imbedding.

b) Let X be a compact space, Y be a Hausdorff space and $g : X \to Y$ be continuous one-to-one map. Show that g is a topological imbedding.

c) Let X and Y be two arbitrary spaces and $f : X \to Y$ be a continuous map. Show that the map $F : X \to X \times Y$ which is given by $F(x) = (x, f(x))$ is a topological imbedding.

4. Let X be a set and $\{(X_\alpha, \tau_\alpha)\}|\alpha \in \Lambda\}$ be a collection of spaces. For each α, let $f_\alpha : X \to X_\alpha$ be a map. Recall that the *weak topology* on X induced by the collection $\{f_\alpha|\alpha \in \Lambda\}$ is the topology τ on X for which the sets $f_\alpha^{-1}(U_\alpha)$ for $\alpha \in \Lambda$ and U_α is open in X_α, form a subbase.

a) Show that τ is the smallest topology on X making each f_α continuous.

b) Let $\{\tau_\alpha|\alpha \in \Lambda\}$ be a family of topologies on a fixed set X and denote by X_α the space consisting of the set X with the topology τ_α. Denote the identity function from X to the space X_α by i_α. Let τ denote the weak topology induced on X by the collection $\{i_\alpha|\alpha \in \Lambda\}$. Exhibit a homeomorphism F from X to the diagonal Δ in the product space $\prod X_\alpha$ and verify that it is indeeed a homeomorphism. (Note: $\Delta = \{x \in \prod X_\alpha | x_\alpha = x_\beta \text{ for all } \alpha, \beta\}$.)