TMS $\sim$ *TOPOLOGY* Examination
February 07, 2019 $\sim$ Start: 10:00
Duration: 3 Hours

In this exam: For each $n \geq 1$, $\mathbb{R}^n$ is equipped with the standard topology, and any subset of $\mathbb{R}^n$ is equipped with the subspace topology induced by the standard topology.

1. Suppose that X and Y are topological spaces, and $X \times Y$ is given the product topology.

(a) Show that the projection map $\pi : X \times Y \to Y$, $\pi(x, y) = y$ is a closed map provided that X is compact.

(b) Show that the projection map $\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\pi(x, y) = y$, is NOT a closed map.

2. Let A_1, A_2 be two subsets of a metric space (X, d). For each $i = 1, 2$, consider the map $f_i : X \to \mathbb{R}$ given by

$$f_i(x) = \inf \{d(x, a) \mid a \in A_i\}.$$

Prove that the map $g(x) = 2f_1(x) - 3f_2(x)$ is continuous using the formal definition of continuity.

3. Consider the product space $\mathbb{R}^\omega = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots$ equipped with the product topology. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of points of $\mathbb{R}^\omega$ given by $X_n = (X_n(1), X_n(2), X_n(3), \cdots)$, so $X_n(k)$ denotes the $k-$th term of the sequence X_n.

(a) Show that X_n converges to a point X in $\mathbb{R}^\omega$ if and only if $X_n(k)$ converges to $X(k)$ for every fixed k.

(b) Is the statement in part (a) valid if $\mathbb{R}^\omega$ is equipped with the box topology? Give a complete verification for your answer.

4. A space X is called *completely regular* if each one point subset of X is closed and whenever $x_0 \in X$ is a point and $A \subseteq X$ is a closed subset not containing x_0, then there is continuous function $f : X \to [0, 1]$ with $f(x_0) = 0$ and $f(A) = \{1\}$.

(a) Show that a connected and completely regular space having at least two points is uncountable.

(b) Let X be a completely regular space, and A and B be disjoint closed subsets of X. If A is compact, then show that there is a continuous map $f : X \to [0, 1]$ so that $f(A) \subset [0, 1/2]$ and $f(B) = \{1\}$.

(c) Find a continuous function $g : [0, 1] \to [0, 1]$ so that $(g \circ f)(A) = \{0\}$ and $(g \circ f)(B) = \{1\}$.