METU - Department of Mathematics
Graduate Preliminary Exam
Algebra - I
September 25, 2017

- Duration: 3 hours.
- Please write your solution for each question on a separate page.
- You can use any statement given below in the solution of any question even if you cannot prove that given statement.

**Question 1.(7+7+8+8 pts.)a)** For a group $G$ and any $g \in G$, define $\phi_g : G \to G$ by $\phi_g(x) = gxg^{-1}$ for all $x \in G$. Show that $Inn(G) = \{\phi_g | g \in G\}$ is a normal subgroup of $Aut(G)$ (the group of automorphisms of $G$).
**b)** Show that $Inn(G)$ is isomorphic to the quotient group $G/Z(G)$ where $Z(G) = \{h \in G | hx = xh \text{ for all } x \in G\}$ is the center of $G$.
**c)** Show that if $G$ has order $p^n$ for some prime $p$ and $n \in \mathbb{Z}^+$, then $Z(G)$ is non-trivial (has more than one element).
**d)** For a $p$-group $G$ as in part (c), if $N$ is a normal subgroup of order $p$ of $G$, then $N \subset Z(G)$.

**Question 2.(7+8+8+7 pts.)a)** Show that if $H$ is a non-abelian group of order $p^3$, then $|Z(H)| = p$ and $Z(H) = H'$ where $H'$ is the commutator subgroup of $H$ generated by all elements of the form $xyx^{-1}y^{-1}$ for $x, y \in H$.
**b)** Let $G$ be a group of order $p^3q^3$ where $p > q > 2$ are primes such that $p$ does not divide $q^2 + q + 1$. Show that $G$ is not simple.
**c)** For $G$ as in part (b), show that $G$ has normal subgroups $N_1$, $N_2$ and $N_3$ such that $N_1 \leq N_2 \leq N_3 \leq G$ and $[G : N_3] = [N_3 : N_2] = [N_2 : N_1] = q$. Is $G$ solvable? Explain.
**d)** Give an example to show that such a group $G$ as in part (b) need not be nilpotent (Hint: use the fact that (without proving it) there exists a non-abelian group $K$ of order 21, and consider the center of $K$).

**Question 3.(10 pts.)** Prove that a finite ring with more than one element and no zero-divisors is a division ring.

**Question 4.(7+7 pts.)** For an ideal $I$ of a commutative ring $R$, the radical of $I$ is defined as $rad(I) = \{r \in R | r^n \in I \text{ for some } n \in \mathbb{Z}^+\}$.
**a)** Show that $rad(I)$ is an ideal of $R$ which contains $I$.
**b)** Show that $rad(I)$ is contained in any prime ideal $P$ of $R$ such that $I \subset P$.

**Question 5.(8+8 pts.)a)** Show that if $f, g \in \mathbb{Z}_p[x]$ ($p$ is a prime) such that $f(x) = g(x^p)$ and $\deg g \geq 1$, then $f$ is reducible in $\mathbb{Z}_p[x]$.
**b)** For a field $F$ explain why $F[x, y]$ is a unique factorization domain and show that it is not a principal ideal domain.