METU - Department of Mathematics
Graduate Preliminary Exam

Algebra I

February, 2009

Duration: 180 min.

1. Given a group G, we can construct a chain $G_1 \xrightarrow{\pi_1} G_2 \xrightarrow{\pi_2} G_3 \xrightarrow{\pi_3} G_4 \xrightarrow{\pi_4} \cdots$, where $G_1 = G$ and $G_{n+1} = \mathrm{Aut}(G_n)$, and $\pi_n(g)(x) = gxg^{-1}$ for all g and x in G_n, for all positive integers n.

a) Show $\pi_n(G_n) \trianglelefteq G_{n+1}$ for all positive integers n.

b) Assuming $C(G) = \langle 1 \rangle$, show that π_n is injective, and $C_{G_{n+1}}(\pi_n(G_n)) = \langle 1 \rangle$ for all positive integers n.

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2. Let G be a finite group and p be the smallest prime divisor of $|G|$. Prove that if H is a subgroup of index p in G, then $H \trianglelefteq G$.

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3. Let Ω be a set, and for each i in Ω, let K_i be a field. Then let R be the ring $\prod_{i \in \Omega} K_i$. Let $\mathfrak{m}$ be a maximal ideal of R. If x is an element $(x_i : i \in I)$ of R, define $S(x) = \{i \in \Omega : x_i \neq 0\}$.

a) Show that, for all x and y in R, if Ω is the *disjoint* union of $S(x)$ and $S(y)$, then exactly one of x and y is in $\mathfrak{m}$.

b) Show that the homomorphism $x \mapsto x/1$ from R to the localization $R_{\mathfrak{m}}$ is surjective.

c) Find the kernel of the homomorphism in (b).

d) What kind of ring is $R_{\mathfrak{m}}$?

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4. Let R be a ring and e be an idempotent in R, that is, $e^2 = e \neq 0$.

a) Show that eRe is a subring of R and e is the identity of eRe

b) Show that if R is finite and contains no nonzero nilpotent elements, then we have $eRe = eR$.