TMS - Complex Analysis
Sep. 2017

Each question is 25 pts.

Notation :
$D = \{z : |z| < 1\}$, $D^* = \{z : 0 < |z| < 1\}$, $\mathbb{C}^* = \mathbb{C} - \{0\}$, $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$.

1. a) Describe all holomorphic functions $f : \Omega \to \mathbb{C}^*$ where

   i) $\Omega = \mathbb{P}^1$, ii) $\Omega = \mathbb{C}$.

   b) Construct a surjective holomorphic map $g : D^* \to \mathbb{C}^*$.

2. Let f be a non-constant entire function.

   a) Show that if there exists $w \in \mathbb{C}$ such that $f(z_n) = w$ for an infinite sequence $\{z_n\}$, then this sequence is unbounded.

   b) Show that if f extends to a meromorphic function on $\mathbb{P}^1$, then there exists an integer $N \geq 1$ such that for any $w \in \mathbb{P}^1$ the set $f^{-1}(w)$ has precisely N elements (counted with multiplicity).

   c) In (b), determine all f such that f' is nowhere 0 in $\mathbb{P}^1$.

3. a) Determine the zeros and the order of zeros of the function $f(z) = e^{z^2} - 1$ in the upper half-plane.

   b) Using part (a), verify that for any z_0 on the unit circle $|z| = 1$ there exists an analytic function $g : D \to \mathbb{C}$ and a sequence $\{z_n\} \subset D$ such that

   - $\lim_{n\to\infty} z_n = z_0$
   - $g(z)$ has a simple zero at each z_n
   - $g(z)$ does not vanish at any other point.

4. Let $f(z)$ be analytic in the punctured disk $D^*(z_0, r)$.

   a) Show that for all $n \geq 1$ and for all closed curves $\Gamma \subset D^*(z_0, r)$, we have

   $$\int_\Gamma f^{(n)}(z)dz = 0.$$

   b) Show that z_0 is not a pole of $g(z) = e^{f(z)}$.