

METU Complex Analysis Preliminary Exam
September 2025

$\mathbb{D} = \{z : |z| < 1\}$ denotes the unit disc in all problems here. Show your work.

1. (25 pts) Find the number of zeros of $f(z) = z^8 + 6z^3 + e^z$ in the annulus $\{1 < |z| < 3\}$.
2. (15+10 pts) Let

$$S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$$

be the vertical strip in the complex plane, and let $f : S \rightarrow \mathbb{D}$ be a holomorphic function such that

$$f\left(\frac{1}{2}\right) = 0.$$

- (a) Construct an explicit conformal map $\phi : S \rightarrow \mathbb{D}$ sending $\frac{1}{2} \mapsto 0$.
- (b) Show that for all $z \in S$,

$$|f(z)| \leq |\phi(z)|.$$

3. (25 pts) Compute $\int_0^{2\pi} \frac{\cos(3\theta)}{5 - 4\cos\theta} d\theta$.

4. (25 pts) Show that if f is a meromorphic function on \mathbb{C} and bounded at infinity (i.e. $\exists R, M > 0$ with $|f(z)| < M$ for all $|z| > R$) then f is a rational function (i.e. $f = \frac{P}{Q}$ for some polynomials P, Q).