20 February 2004

Graduate Preliminary Examination
Geometry
Duration: 3 hours

1. Let S^2 be the unit circle in $\mathbb{R}^3$. Considering S^2 oriented by outer normal field

a) exhibit a positively oriented basis of the tangent space for each point of S^2,

b) determine whether the reflection $F : S^2 \to S^2$ which is given by $F(x, y, z) = (x, -y, z)$ is orientation preserving or not.

2. Let X, Y be smooth vector fields on a smooth manifold M. Then XY defined by $(XY)(f) = X(Yf)$ makes sense as a smooth operator. We know that $[X, Y] = XY - YX$ is a smooth vector field.

a) Show that $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$ for all smooth real valued functions f and g on M.

b) Let $(U; x_1, \cdots, x_n)$ be a coordinate neighborhod on M and let $\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\}$ be the associated coordinate frames. Show that $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$ for each i, j with $1 \leq i \leq n$, $1 \leq j \leq n$.

c) Assuming that $\dim M = 2$, compute the components of $[X, Y]$ in terms of the components of X and Y with respect to a coordinate neighborhood.

3. Let $F : M \to N$ be a smooth map, $q \in N$ a regular value and $L = F^{-1}(q) \subset M$. Show that for any $p \in L$ the tangent space T_pL is the kernel of the induced map $F_* : T_pM \to T_qN$.

4. Let w be the 2-form on $\mathbb{R}^3 \setminus (0, 0, 0)$ given by $w = d(\frac{1}{x^2+y^2+z^2} dy)$.

a) Find the local expression of the pull back of w on M with respect to the local parametrization

$$
\begin{aligned}
x &= 2\cos u\,(1+\cos v)-2\\
y &= 2\sin u\,(1+\cos v)\\
z &= \sin v \qquad\qquad u, v \in (0, 2\pi).
\end{aligned}
$$

b) Find $\displaystyle\int_M w$.