**Differentiable Manifolds**
**TMS EXAM**
13 February 2015

**Duration: 3 hr.**

**1.** Let $S^2$ be the unit sphere in $\mathbb{R}^3$. Consider it with the topology relative to $\mathbb{R}^3$. Let $i : S^2 \to \mathbb{R}^3$ be the inclusion map.

(**a**) Show that $i$ is an immersion.

(**b**) Is $i$ an embedding? Why?

**2.** Let $M, N$ be two differentiable manifolds and $f : M \to N$ be a smooth map. Define a new map $F : M \to M \times N$ by $F(p) = (p, f(p))$.

(**a**) Show that $F$ is smooth.

(**b**) Show that $F_*(v) = (v, f_*(v))$ where $F_*$ and $f_*$ are induced maps at a point $p$ of $M$ and $v$ is a tangent vector of $M$ at $p$.

(**c**) Show that the tangent space to graph(f) at the point $(p, f(p))$ is the graph of $f_* : T_pM \to T_{f(p)}N$

**3.** Consider the 1-form $w = (x^2 + 7y)dx + (-x + y\sin \mathrm{y}^2)\mathrm{dy}$ on $\mathbb{R}^2$.

(**a**) Is $w$ exact? Is it closed?

(**b**) Compute the integral of $w$ over each side of the triangle whose vertices are $(0,0), ((1,0), (0,2)$ where the sides are oriented in such a way that the triangle is oriented counterclockwise.

**4.** Let F: $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the map $F(p) = -p$.

(**a**) What is the induced map $F_*$? Why?

(**b**) Show that antipodal map $A : S^n \to S^n$ which is the restriction of $F$ on the $n$-sphere is orientation preserving if and only if $n$ is odd.

(**c**) Prove that the real projective space $\mathbb{R}P^n$ is orientable if and only if $n$ is odd.