PRELIMINARY EXAM PROBLEMS
Differential Equations (ODE), 3 hours, 2017/1

1. Supppose that $x_1(t)$ and $x_2(t)$ are the solutions of $x' + c(t)x = 0$ with $x_1(t_1) = a, x_2(t_2) = b$, where a, b are constants and t_1, t_2, t are members of an interval $I \subset R$, and $c(t)$ is a continuous function. Solve the equation and show that $x_1(t) - x_2(t) \to 0$ as $t_1 \to t_2$ and $a \to b$ for all $t \in I$.

2. Consider the IVP
$$x' = t^2 + x^2, x(0) = 0, 0 \leq t \leq a, |x| < b.$$
Show that

   (i) the solution exists on $0 \leq t \leq \min(a, \frac{b}{a^2+b^2})$;

   (ii) the maximum value of $\frac{b}{a^2+b^2}$ is $1/(2a)$ for a fixed a;

   (iii) $h = \min(a, 1/(2a))$ is largest when $a = 1/\sqrt{2}$;

   (iv) discuss the maximum interval of existence on the basis of (ii) and (iii).

3. Solve the BVP,
$$y'' + y = 0, y(0) = 0, y(a) = y_0. \tag{1}$$

4. Consider the following scalar equation
$$x' = a(t)x, \tag{2}$$
where $a(t) : R \to R$ is a continuous function. Prove that the zero solution, $x \equiv 0$, of the equation is uniformly stable if and only if
$$\int_{t_0}^{t} a(s)ds \leq M < \infty, t \geq t_0 > 0,$$
with M constant.