METU MATHEMATICS DEPARTMENT
Real Analysis
September 2017 - TMS

1. Let (E, μ) be a finite measure space. Let $\{f_n\}$ be a sequence of measurable complex valued functions such that $\forall x \in E$, $\lim f_n(x) = f(x)$ exists.

a) Show that $\forall \epsilon > 0$ $\forall \delta > 0$, $\exists N$, and a measurable set F with $\mu(F) < \delta$ such that.

$$|f_n(x) - f(x)| < \epsilon \;\; \forall n \geq N \;\; \forall x \in E \setminus F.$$

b) Is the above result true if $\mu(E) = \infty$?

2. Calculate $\displaystyle\int_0^\infty \frac{\sin x}{x} dx$.

(Hint: Observe that $\forall R > 0$, $\int_0^R \frac{\sin x}{x} dx = \displaystyle\int_0^R \int_0^\infty \sin x \;\; e^{-xt} dt dx$ and use Fubini-Tonelli Theorems)

3. Let f be a bounded, real-valued, measurable function on $[0,1]$, s.t. $\displaystyle\int x^n f dm = 0$ for $n = 0, 1, 2, \cdots$ where m denotes the Lebesgue measure.

Show that $f(x) = 0$ a.e.

4. Let $f : [0,1] \to \mathbb{R}$ be Lebesgue-measurable.

a) Prove that if $\displaystyle\int_0^1 f(x) dx$ (in Riemann sense) exists then $f \in L^1([0,1])$.

b) Is the converse true? Prove or disprove (by giving an example).