Real Analysis

TMS

20.09.2019

1) a) Is the following true or false? Explain. "There is a subset A of $\mathbb{R}$ which is **not** measurable, but such that $B = \{x \in A : x \text{ is irrational } \}$ is measurable."

b) Compute the Lebesgue integral of the function

$$f(x,y) = \begin{cases} 1 \quad \text{if} \quad xy \quad \text{is rational} \\ \\ 0 \quad \text{if} \quad xy \quad \text{is irrational} \end{cases} \qquad \text{over the square} \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1.$$

2) Find the limit $\lim\limits_{n\to\infty} \int_0^1 \cos(x^n)dx$. Explain.

3) Prove that $\int_0^1 \sqrt{x^4 + 4x^2 + 3}dx \leq \frac{2}{3}\sqrt{10}$.
Hint: Factorize $x^4 + 4x^2 + 3$ first.

4) Suppose $\{f_n\}$ is a sequence of measurable functions on $[0,1]$ such that $\lim\limits_{n\to\infty} \int_0^1 |f_n| = 0$ and that there is an integrable function g on $[0,1]$ with $|f_n|^2 \leq g \forall n$. Prove that $\lim\limits_{n\to\infty} \int_0^1 |f_n|^2 = 0$ state the theorems you used.