TOPOLOGY TMS EXAM
October 02 2015

Duration: 3 hours.

(1) Let $f\colon X \to Y$ be a map between topological spaces. Show that the following are equivalent.

(i) f is continuous and open.

(ii) $f^{-1}(\mathrm{Int}(B)) = \mathrm{Int}(f^{-1}(B))$ for all $B \subset Y$.

(iii) $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$ for all $B \subset Y$.

(2) Let (x_n) be a sequence of points of the space $\prod_{\alpha \in J} X_\alpha$ endowed with the product topology. Let $\pi_\beta\colon \prod_{\alpha \in J} X_\alpha \to X_\beta$ be the projection mapping associated with the index β.
Show that the sequence (x_n) converges to x if and only if the sequence $\pi_\beta(x_n)$ converges to $\pi_\beta(x)$ for each $\beta \in J$.
Is this true if you use the box topology instead of the product topology?

(3) Let X be the subspace of $\mathbb{R}^2$ defined by $X = \cup_{n=1}^{\infty} C_n$, where $C_n = \{(x,y) : (x - 1/n)^2 + y^2 = 1/n^2\}$ (X is the union of the circles with center $(1/n, 0)$ and radius $1/n$ for $n = 1, 2, 3, \ldots$) Let Y be the quotient space formed by starting with $\mathbb{R}$ and defining $x \sim y$ if either $x = y$ or if $x, y \in \mathbb{Z}$. Prove that X and Y are not homeomorphic.

(4) Let X be metrizable. Show that the following are equivalent.

(i) X is bounded under every metric that gives the topology of X.

(ii) Every continuous function $\phi\colon X \to \mathbb{R}$ is bounded.

(iii) X is limit point compact.