Conics on polarized $K3$-surfaces

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Generalizing Barth and Bauer, denote by $N_{2n}(d)$ the maximal number of smooth degree d rational curves that can lie on a smooth $2n$-polarized $K3$-surface X in $\mathbb{P}^{n+1}$. Originally, the question was raised in conjunction with smooth quartics in $\mathbb{P}^3$, which are $K3$-surfaces.

The numbers $N_{2n}(1)$ are well understood, whereas the only known value for $d = 2$ is $N_6(2) = 285$. I will discuss my recent discoveries that support the following conjecture on the conic counts in the remaining interesting degrees.

Conjecture. One has $N_2(2) = 8910$, $N_4(2) = 800$, and $N_8(2) = 176$.

The approach used does not distinguish (till the very last moment) between reducible and irreducible conics. However, extensive experimental evidence suggests that all conics are irreducible whenever their number is large enough.

Conjecture. There exists a bound $N^*_{2n}(2) < N_{2n}(2)$ such that, whenever a smooth $2n$-polarized $K3$-surface $X \subset \mathbb{P}^{n+1}$ has more than $N^*_{2n}(2)$ conics, it has no lines and, in particular, all conics on X are irreducible.

We know that $249 \leq N^*_6(2) \leq 260$ is indeed well defined, and it seems feasible that $N^*_2(2) \geq 8100$ and $N^*_4(2) \geq 720$ are also defined; furthermore, conjecturally, the lower bounds above are the exact values (attained at surfaces maximizing the number of lines).