M.E.T.U

Department of Mathematics

Preliminary Exam - Sep. 2011

ALGEBRA I

Duration : 180 min.

Each question is 25 pt.

1. Let G be a finite group.

   a) Suppose that M and N are normal subgroups of G such that both G/M and G/N are solvable. Prove that $G/M \cap N$ is solvable.

   b) Prove that G has a subgroup L which is the unique smallest subgroup with the properties of being normal with solvable quotient.

   c) Suppose that G has a subgroup H isomorphic to A_5. Show that $H \subseteq L$.

2. Let G be a group.

   a) Prove that if N is a normal subgroup of G then $G/C_G(N)$ is isomorphic to a subgroup of the group $Aut(N)$ of all automorphisms of N.

   Prove the following assuming that G is a finite group with

   $$\gcd(|G|, |\mathrm{Aut}(G)|) = 1.$$

   b) G is abelian.

   c) Every Sylow subgroup of G is cyclic of prime order.

   d) G is a cyclic group of squarefree order such that if p and q are prime divisors of $|G|$, then we have $p \not\equiv 1(\mathrm{mod} q)$ and $q \not\equiv 1(\mathrm{mod} p)$.

3. Let $\mathbb{F}_2$ be the field with 2 elements and $R = \mathbb{F}_2[X, 1/X]$ for an indeterminate X.

   Prove the following:

   a) The unit group of R is generated by X.

   b) There are infinitely many distinct ring endomorphisms of R.

   c) The ring automorphism group $\mathrm{Aut}(R)$ is of order 2.

4. Let $(R, +, \cdot)$ be a commutative ring with identity $1 \neq 0$, D be a multiplicative set in R ($1 \in D$), $D^{-1}T = \{\frac{t}{d} \,|\, t \in T,\ d \in D\}$, $S = D^{-1}R$ be the ring of fractions of R with respect to D and $\pi : R \longrightarrow S$ be the ring homomorphism given by $\pi(r) = \frac{r}{1}$.

   Consider the following property for **proper** ideals I of R:
   (*) if $xr \in I$ for some $r \in D$ then $x \in I$.

   a) Show that if Q satisfies (*) and $D = R \backslash Q$ then

   $$\ker(\pi) \subseteq Q \cap \text{Zerodivisors(R)}.$$

   b) Show that if Q satisfies (*) and $D^{-1}Q = D^{-1}J$ then $J \subseteq Q$.

   c) Show that if Q satisfies (*) and $D^{-1}Q$ is a prime ideal then $D \cap Q = \emptyset$ and Q is a prime ideal.

   d) Show that if P is a prime ideal and $D \cap P = \emptyset$ then P satisfies (*) and $D^{-1}P$ is a prime ideal.