METU MATHEMATICS DEPARTMENT
GRADUATE PRELIMINARY EXAMINATION
ALGEBRA I, SEPTEMBER 2015

SEPTEMBER 28, 2015

1.a) Show that the alternating group A_n has no proper subgroup of index less than n, provided that $n \geq 5$. (Hint: Assume that such a subgroup H exits and then consider the action of A_n on the left cosets of H in A_n.)

b) Use Part (a) to prove that S_n has no proper subgroup of index less than n other than A_n, provided that $n \geq 5$.

2.a) Let p, q be primes with $p \geq q^2$ and G be group of order pq^2. Prove that G has a normal Sylow p-subgroup.

b) In addition to the assumptions in Part (a) assume further that the greatest common divisor $(q^2, p-1) = 1$. Show that the group in Part (a) is abelian.

3.a) Show that the map

$$f : \mathbb{C} \longrightarrow M_2(\mathbb{R}), \qquad f(a+ib) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

is an injective ring homomorphism, where $M_2(\mathbb{R})$ is the ring of $2x2$-real matrices.

b) Is the image an ideal in $M_2(\mathbb{R})$? Why?

4) Let R be the ring of continuous functions on the interval $[0,1]$.

a) What are the units of the ring R?

b) For any $a \in [0,1]$, let I_a be the set of elements $f \in R$ with $f(a) = 0$. Show that I_a is a maximal ideal in R.

c) Show that the set of elements $f \in R$ with $f(1/2) = 0 = f(1/4)$ is an ideal. Is it prime?

d) Show that any maximal ideal of R is of the form I_a, for some $a \in [0,1]$.