By signing below, I pledge that I will write this examination as my own work and without the assistance of others or the usage of unauthorized material or information. I understand that possession of any kind of electronic device during the exam is prohibited. I also understand that not obeying the rules of the examination will result in immediate cancellation and disciplinary procedures.

METU Department of Mathematics

<table>
<tr><td colspan="3" align="center">Graduate Preliminary Exam</td></tr>
<tr><td>Acad. Year : 2023
Semester : Fall
:
Date : September 29, 2023
Time : 10:00
Duration : 180 minutes</td><td colspan="2">Full Name (USE CAPITAL LETTERS):

Student ID:
</td></tr>
<tr><td colspan="2" align="center">5 QUESTIONS
TOTAL 100 POINTS</td><td>Signature:</td></tr>
</table>

Q1 (20 *pts*) Suppose G is a finite group and f is an automorphism of G fixes more than half of the elements of G. Show that f is the identity automorphism.

Q2 (20 *pts*) Prove that if an infinite group G contains a proper subgroup of finite index, then G contains a proper normal subgroup of finite index.

Q3 (20 *pts*) Show that any group of order 154 is solvable.

Q4 (20 *pts*) Let K be a field. Prove that the polynomial ring $K[x]$ has infinitely many maximal ideals.

Q5 (20 *pts*) Suppose R is a ring with identity 1_R. An element $e \in R$ is called idempotent if $e^2 = e$. Assume e is an idempotent in R and $er = re$ for all $r \in R$.

(a) Show that Re and $R(1_R - e)$ are two-sided ideals R.

(b) Show that $R \simeq Re \times R(1_R - e)$.

(c) Show that e and $1_R - e$ are identities for the subrings Re and $R(1_R - e)$ respectively.