Q1 (20 pts) Suppose $G$ is a finite group and $f$ is an automorphism of $G$ fixes more than half of the elements of $G$. Show that $f$ is the identity automorphism.

Q2 (20 pts) Prove that if an infinite group $G$ contains a proper subgroup of finite index, then $G$ contains a proper normal subgroup of finite index.

Q3 (20 pts) Show that any group of order 154 is solvable.

Q4 (20 pts) Let $K$ be a field. Prove that the polynomial ring $K[x]$ has infinitely many maximal ideals.

Q5 (20 pts) Suppose $R$ is a ring with identity $1_R$. An element $e \in R$ is called idempotent if $e^2 = e$. Assume $e$ is an idempotent in $R$ and $er = re$ for all $r \in R$.

(a) Show that $Re$ and $R(1_R - e)$ are two-sided ideals $R$.

(b) Show that $R \simeq Re \times R(1_R - e)$.

(c) Show that $e$ and $1_R - e$ are identities for the subrings $Re$ and $R(1_R - e)$ respectively.