METU Department of Mathematics
PRELIMINARY EXAM
Algebra - I
Feb 13, 2017

Name: ID Number: Signature:

Duration is 3 hours.
Please write your solutions for each question on a separate page.

1.(10+10+5=25 pts.) Let G be a finite group of order $p^n q^2$ where $p > q$ are odd primes and $n \geq 1$.
a) Show that G has a normal subgroup N of index q^2 ($[G : N] = q^2$).
b) Show that G has a normal subgroup K of index q.
c) What are the factors of a composition series of G?

2.(9+8+8=25 pts.) For a group G, the commutator $[x, y]$ of two elements x and y of G is defined as $[x, y] = xyx^{-1}y^{-1}$, and the commutator subgroup G' of G is the subgroup generated by all commutators in G:

$$G' = \langle \{ [x, y] \in G \,|\, x \in G, y \in G \} \rangle.$$

The derived series of G is the sequence of subgroups:

$$G > G^{(1)} > G^{(2)} > \ldots > G^{(i)} > \ldots$$

where $G^{(1)} = G'$ and $G^{(i+1)} = (G^{(i)})'$ for $i \geq 1$.
a) Show that for any automorphism f of G we have $f(G^{(i)}) = G^{(i)}$ for any $i \geq 1$.
b) Use part (a) to show that $G^{(i)}$ is a normal subgroup of G for all $i \geq 1$.
c) Show that S_n is not solvable for $n \geq 5$ by explicitly writing down the derived series of S_n.

3.(5+10+10=25 pts.) Let R be a commutative ring with identity 1.
a) Let $M_2(R)$ be the ring of all 2×2 matrices with entries in R. Show that for any ideal I of R, $M_2(I) = \{A \in M_2(R) \,|\, A_{ij} \in I \text{ for all } i, j\}$ is an ideal of $M_2(R)$.
b) Show that any ideal J of $M_2(R)$ is of the form $J = M_2(I)$ for some ideal I of R.
c) Show that if I is an ideal of R, then any prime ideal of the quotient ring R/I is of the form P/I for some prime ideal P of R.

4.(9+8+8=25 pts.) a) Show that the polynomial ring $\mathbb{Z}[x]$ is a unique factorization domain which is not a principal ideal domain.
b) Let R be a commutative ring with identity 1. Show that if $u \in R$ is a unit and $r \in R$ is a nilpotent element of R (which means $r^n = 0$ for some $n \in \mathbb{Z}^+$), then $u + r$ is a unit in R.
c) Use part (b) to show that if a_0 is a unit in R and a_i is a nilpotent element of R for each $1 \leq i \leq d$, then $f = \sum_{i=0}^{d} a_i x^i$ is a unit in the polynomial ring $R[x]$.