M E T U Department of Mathematics

Graduate Preliminary Exam, Algebra I	Spring 2019	06.02.2019	10:00
Last Name : Name : Student No :	Signature :		
5 QUESTIONS			TOTAL 100 POINTS
1 | 2 | 3 | 4 | 5	Duration 3 hours		

(1) Let p be a prime number. Let G be an **infinite** group such that for any subgroup $\{e\} \neq H \subsetneqq G$ $|H| = p$. (i.e., any non-trivial proper subgroup of G has order p). (There are such groups!).

(a) (10 pts) Show that G can be generated by two elements (i.e., there exist $a, b \in G$ such that $G = \langle a, b \rangle$.)

(b) (10 pts) Show that G is simple.

(2) (a) (10 pts) Show that any group of order 12 is solvable.

(b) (10 pts) Show that any group of order $588 = 7^2 \cdot 3 \cdot 2^2$ is solvable.

(3) (20 pts) Recall the following:

Theorem: If F is a free abelian group of rank n then any subgroup of F is free abelian of rank at most n

Let G be an abelian group which is generated by n elements. Show that every subgroup of G can be generated by at most n elements.

(4) Recall the following:

- A ring R is called *Noetherian* if for any ascending sequence $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ of ideals of R, there exists n such that $I_n = I_{n+1} = I_{n+2} = \ldots$
- Let R be a commutative ring with identity. For $r \in R$ let $Ann(r) = \{s \in R \mid sr = 0\}$. (It is clear that $Ann(r)$ is an ideal of R.)

Let R be a commutative ring with identity which is also Noetherian.

(a) (5 pts) Show that $\{Ann(r) \mid r \neq 0\}$ has a maximal element (with respect to inclusion).

(b) (10 pts) Show that there exists $r \in R$ such that $Ann(r)$ is a prime ideal.

(c) (5 pts) If $Ann(r)$ is prime and $s \in R$, show that $sr = 0$ or $Ann(sr)$ is prime.

(5) Let $\mathbb{Z}[2i] = \{a + 2bi \mid a, b \in \mathbb{Z}\}$. Clearly $\mathbb{Z}[2i]$ is a subring of $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$.

(a) (10 pts) Show that $\mathbb{Z}[2i]$ is **not** a unique factorization domain.

(b) (10 pts) Prove or disprove: If R is a principal ideal domain then every subring of R is a principal ideal domain.