M E T U Department of Mathematics

Graduate Preliminary Exam, Algebra I	Fall 2021	March 4, 2022
Last Name : Name : Student No :	Signature :	
4 QUESTIONS		TOTAL 100 POINTS

1	2	3	4		Duration 3 hours		

(1) Let $\{G_i \mid i \geq 1\}$ be a family of finite groups and let G be the weak (restricted) direct product of this family .

(a) (15 Points) Show that every finitely generated subgroup of G is finite.

(b) (10 Points) Show that this may not be true for the direct product of such a family.

(2) (a) (5 Points) Let G be a group and $H \subseteq C(G)$ be subgroup of G. Show that if G/H is cyclic then G is abelian. (Here, $C(G)$ denotes the center of the group G.)

(b) (10 Points) Show that if $Aut(G)$ is cyclic, then G is abelian. (Here, $Aut(G)$ denotes the automorphism group of G).

(c) (10 Points) Give an example of an abelian group G such that $Aut(G)$ is not abelian.

(3) (a) (10 Points) Let R be a principal ideal domain. Show that there is no infinite chain of ideals such that

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$$

(b) (15 pts) Let R be a unique factorization domain. Show that for any infinite chain of principal ideals of R,

$$(a_1) \subseteq (a_2) \subseteq (a_3) \cdots$$

there is $n \geq 1$, such that $(a_i) = (a_n)$ for $i \geq n$.

(4) Let $R = \{f(x) \in \mathbb{Q}[x] \mid f(0) \in \mathbb{Z}\} \subseteq \mathbb{Q}[x]$.

(a) (5 Points) Show that R is an integral domain.

(b) (10 Points) Consider the following ideals of R:

$$I = \{f(x) \in R \mid f(0) = 0\}, \quad J = (x), \quad K = (2)$$

Which of I, J, K are prime/maximal ideals? Explain.

(c) (10 Points) Is R a principal ideal domain? Explain.