M E T U Department of Mathematics

ALGEBRA I PRELIMINARY EXAM FEBRUARY 2024		
F U L L N A M E (**write in CAPITAL letters**)	S T U D E N T I D	DURATION 180 MINUTES
4 QUESTIONS		TOTAL 100 POINTS

1. (6+7+7 Points) A group G is a "nice group" if

1) G is abelian,

2) For every subgroup $H \leq G$ there is a subgroup $K \leq G$ such that $G = H \oplus K$.

a) If G is a nice group, show that every subgroup of G is nice.

b) If G is a nice group show that every element of G has finite order.

c) If G is a nice group and p is a prime, show that G has no elements of order p^2.

2. (10+10 Points) Suppose a finite group G acts on a finite set X.
For $g \in G$ let $Fix(g) = \{x \in X \mid g.x = x\}$. Burnside's Lemma states the following:

$$\text{The number of orbits of the action } = \frac{1}{|G|} \sum_{g \in G} |Fix(g)|$$

(You do not need to prove Burnside's lemma)

a) Prove that if G acts transitively on X and $|X| > 1$, then there is $g \in G$ such that $Fix(g)$ is empty.

b) Let G be a finite group and H be a proper subgroup of G. Show that G is not a union of conjugates of H.

3. (10+10 Points) Consider $\mathbb{Q}$ and $\mathbb{Z}$ as additive groups.

a) Find all group homomorphisms from $\mathbb{Z}$ to $\mathbb{Q}$.

b) Find all group homomorphisms from $\mathbb{Q}$ to $\mathbb{Z}$.

4. (10+10 Points) Let R be a ring with identity. Show that for an element $r \in R$, r belongs to every maximal ideal of R if and only if $1 - rx$ is a unit for every $x \in R$.

5. (10 Points) Let R be commutative ring such that for any ideals $I, J \subseteq R$, either $I \subseteq J$ or $J \subseteq I$. Show that a finitely generated ideal of R must be principal.

6. (10 Points) Let R be a commutative ring with identity and let U be maximal among non-principal ideals of R. Show that U is a prime ideal.