METU Mathematics Department
Graduate Preliminary Examination
Algebra I, February 2025

1. Suppose that H is a subgroup of G with finite index n. Prove that there is a normal subgroup K of G with $K \leq H$ and $[G : K] \leq n!$.

2. Let G be a group of order 105.

   (a) Show that G is not simple.

   (b) Show that G is solvable.

3. Let H be a subgroup of a finite Abelian group A. Show that there is a subgroup K of A such that $A/K \cong H$.

4. Let R be a commutative ring with identity 1_R. If $R[x]$ is a principal ideal domain then show that R is a field.

5. Let $R = \mathbb{Z}[\sqrt{-5}]$, a subring of $\mathbb{C}$. Consider the ideals $I = (3, 1 + \sqrt{-5})$ and $J = I^2$.

   (a) Show that I is not a principal ideal.

   (b) Show that J is a principal ideal and find a generator.

   (c) Show that $|R/J| = 9$. Is the quotient ring R/J isomorphic to one of the rings $R_1 = \mathbb{Z}_9$ or $R_2 = \mathbb{Z}_3 \times \mathbb{Z}_3$? Justify your answer.