Graduate Preliminary Examination

Algebra II

18.2.2005: 3 hours

Problem 1. Prove or give a counter-example to the following statement: If M/L and L/K are algebraic extensions of fields, then M/K is algebraic.

Problem 2. Let p be a prime and let $GF(p^m)$ denote the finite field of order p^m.

(a) Show that for any positive integer m, there exists a finite field of order p^m

(b) Show that if $GF(p^m)$ is isomorphic to a subfield of $GF(p^n)$, then m divides n.

(c) Let E be the algebraic closure of $GF(p)$. Show that there is an intermediate field L between $GF(p)$ and E with $|L : GF(p)| = \infty$ and $|E : L| = \infty$

Problem 3. Let R be a commutative ring with identity , I an ideal of R and $L = \{a \in R : \ aI = 0\}$

(a) Prove that each $a \in L$ induces an R-module homomorphism

$$\bar{\lambda}_a : R/I \to R$$

(b) Using (a), prove that the R-modules L and $Hom_R(R/I, R)$ are isomorphic.

Problem 4. Let R be a commutative ring with identity, and let M be a unitary R-module. Then M is called:

- **torsion-free**, if $r \cdot m = 0$ implies either $r = 0$ or $m = 0$ where r in R and m in M;
- **divisible**, if for all m in M and non-zero r in R, there is n in M such that $r \cdot n = m$.

Assume M is torsion-free and non-trivial.

(a) Prove that R is an integral domain. Show that, the hypothesis that M is non-trivial is necessary.

Now let K be the quotient-field of R.

(b) Prove that $K \otimes_R M$ is torsion-free as a K-module.

(c) Prove that, if M is divisible, then $\phi : m \mapsto 1 \otimes m$ is an R-module epimorphism from M onto $K \otimes_R M$.