**METU - Department of Mathematics**
**Graduate Preliminary Exam**

**Algebra II**

February, 2009

**Duration:** 180 min.

**1.** Let $\alpha$ be the real positive 16th root of 3 and consider the chain of intermediate fields

$$\mathbb{Q} \subseteq \mathbb{Q}(\alpha^8) \subseteq \mathbb{Q}(\alpha^4) \subseteq (\alpha^2) \subseteq \mathbb{Q}(\alpha) = F.$$

a) Compute the degrees of these five intermediate fields over $\mathbb{Q}$ and conclude that these fields are all distinct.

b) Show that every intermediate field between $\mathbb{Q}$ and $F$ is one of the above. (Hint: If $\mathbb{Q} \subseteq K \subseteq F$, consider the constant term of the minimal polynomial of $\alpha$ over $K$).

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**2.** Let $p$ be a prime number and let $w_p = e^{2\pi i/p}$ be the $p$th root of 1 in $\mathbb{C}$.

a) Show that Gal $(\mathbb{Q}(w_p)/\mathbb{Q})$ is isomorphic to the multiplicative group $\mathbb{Z}_p^*$.

b) Let $F$ be a field containing $w_p$ and let $a$ be an element of $F$ which is not the $p$th power of any element of $F$. Show that if $E$ is the splitting field of the polynomial $x^p - a \in F[x]$, then Gal $(E/F)$ is isomorphic to the additive group $\mathbb{Z}_p$.

c) If $K$ is the splitting field of $x^p - 2 \in \mathbb{Q}[x]$, show that $|K : \mathbb{Q}| = p(p-1)$.

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**3.** Let $R$ be a ring. Recall that an $R$-module $P$ is called projective if for every $R$-module epimorphism $f : A \to B$ and every $R$-module homomorphism $g : P \to B$, there exists an $R$-module homomorphism $h : P \to A$ such that $fh = g$.

a) Let $P$ be an $R$-module for a ring $R$. Show that if there is a free $R$-module $F$ and an $R$-module $K$ such that $F \cong K \oplus P$, then $P$ is projective. (You may use the fact that every free module is projective).

b) Let $R$ be a commutative ring. Suppose that $R$-modules $P$ and $Q$ are projective. Show that $P \otimes_R Q$ is projective.

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**4.** Let $R$ be a ring with unity and suppose that $R$ can be written as the sum $R = \sum_{i=1}^{m} I_i$, where $I_i$ are finitely many (two-sided) ideals of $R$ satisfying $I_i \cap I_j = 0$ whenever $i \neq j$.

a) Prove that, for every simple right $R$-module $M$, there exists a unique subscript $k$ such that $MI_k \neq 0$

b) Show that if $i \neq j$, then every right $R$-module homomorphism $\theta : I_i \to I_j$ is the zero map.