

## GRADUATE PRELIMINARY EXAMINATION

### ALGEBRA II Spring 2010

1. Let  $R$  be a commutative ring with identity 1 and let  $Q$  be an injective  $R$ -module.

If  $L \xrightarrow{\alpha} M \xrightarrow{\beta} N$  is an exact sequence of  $R$ -modules and  $R$ -homomorphisms with the property that  $f \circ \alpha = 0$  for an  $R$ -homomorphism  $f : M \rightarrow Q$ , show that there is an  $R$ -homomorphism  $g : N \rightarrow Q$  with  $g \circ \beta = f$ .

2. A nonzero left module  $M$  (over some ring) is called

• **simple**, if  $M$  has no proper nonzero submodule;

• **complemented**, if every submodule of  $M$  is a direct summand of  $M$  (that is, for every submodule  $A$  of  $M$ , there is a submodule  $B$  of  $M$  such that  $M = A \oplus B$ , which means  $M = A + B$  and  $A \cap B = 0$ ).

(a) Give an example of a simple module.

(b) Give an example of a complemented module that is not simple.

(c) Show that every nonzero submodule of a complemented module is complemented.

(d) Show that every complemented module has a simple submodule.

3. Suppose  $K, L$ , and  $M$  are fields, and  $K \subseteq L \subseteq M$ . Prove or disprove the following statements.

(a) If  $M/L$  and  $L/K$  are normal, then so is  $M/K$ .

(b) If  $M/K$  is normal, then so is  $M/L$ .

(c) If  $M/L$  is normal, then so is  $M/K$ .

(d)  $(K, +) \not\cong (K^*, \cdot)$ .

4. Consider the polynomial  $f(x) = x^5 - 6x + 3 \in \mathbb{Q}[x]$

(a) Using Eisenstein's criterion, prove that  $f$  is irreducible over  $\mathbb{Q}$ .

(b) Let  $E$  be the splitting field of  $f$ . Show that there exists  $\sigma \in \text{Gal}(E/\mathbb{Q})$  of order 5.

(c) Prove the following:

There exists  $\tau \in \text{Gal}(E/\mathbb{Q})$  of order 2 and hence  $\text{Gal}(E/\mathbb{Q}) \cong S_5$ .

(Hint : You may assume that  $f(x)$  has exactly one pair of complex conjugate roots.)

(d) Is  $f(x)$  solvable by radicals over  $\mathbb{Q}$ ? Why?