GRADUATE PRELIMINARY EXAMINATION

ALGEBRA II
Spring 2010

1. Let R be a commutative ring with identity 1 and let Q be an injective R-module. If $L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is an exact sequence of R-modules and R-homomorphisms with the property that $f \circ \alpha = 0$ for an R-homomorphism $f : M \longrightarrow Q$, show that there is an R-homomorphism $g : N \longrightarrow Q$ with $g \circ \beta = f$.

2. A nonzero left module M (over some ring) is called
• **simple**, if M has no proper nonzero submodule;
• **complemented**, if every submodule of M is a direct summand of M (that is, for every submodule A of M, there is a submodule B of M such that $M = A \oplus B$, which means $M = A + B$ and $A \cap B = 0$).

(a) Give an example of a simple module.

(b) Give an example of a complemented module that is not simple.

(c) Show that every nonzero submodule of a complemented module is complemented.

(d) Show that every complemented module has a simple submodule.

3. Suppose K, L, and M are fields, and $K \subseteq L \subseteq M$. Prove or disprove the following statements.

(a) If M/L and L/K are normal, then so is M/K.

(b) If M/K is normal, then so is M/L.

(c) If M/L is normal, then so is M/K.

(d) $(K, +) \ncong (K^*, \cdot)$.

4. Consider the polynomial $f(x) = x^5 - 6x + 3 \in \mathbb{Q}[x]$

(a) Using Eisenstein's criterion, prove that f is irreducible over $\mathbb{Q}$.

(b) Let E be the splitting field of f. Show that there exists $\sigma \in Gal(E/Q)$ of order 5.

(c) Prove the following:

There exists $\tau \in Gal(E/Q)$ of order 2 and hence $Gal(E/Q) \cong S_5$.

(Hint : You may assume that $f(x)$ has exactly one pair of complex conjugate roots.)

(d) Is $f(x)$ solvable by radicals over $\mathbb{Q}$? Why?