TMS EXAM
February 16, 2012
ALGEBRA II

1. Prove that there is no field F such that $F^+ \cong F^*$.

2. Let F be a field with 16 elements.

   (i) Show that there exists an element $\alpha \in F$ with $\alpha^4 = \alpha + 1$.

   (ii) Find the factorization of $x^3 + x + 1 \in F[x]$ into irreducible polynomials over F.

   (iii) Find all subfields of F.

   (iv) Does there exist a quadratic, irreducible polynomial over F ? Explain your reasoning.

3. Let R be a commutative ring with unity, and let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

   be an exact sequence of R-modules. Let $r, s \in R$ be such that $(r, s) = R$. Suppose that $rA = sC = 0$.

   (i) Show that the map $\alpha : C \longrightarrow C$ given by $\alpha(c) = rc$ is an isomorphism.

   (ii) Show that $g|_{rB}$ is an isomorphism between rB and C.

   (iii) Show that $B \cong A \oplus C$.

4. Let R and S be two rings, A a right R-module, C a right S-module and B an (R, S)-bimodule. Show that $\mathrm{Hom}_S(A \otimes_R B, C)$ and $\mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$ are isomorphic abelian groups.