M E T U Department of Mathematics

Graduate Preliminary Exam, Algebra II Spring 2022 March 4, 2022		
Last Name : Name : Student No :	Signature :	
4 QUESTIONS		TOTAL 100 POINTS
1 | 2 | 3 | 4	**Duration 3 hours**	

(1) (25 Points) If $T : V \to V$ is a linear transformation on a vector space V over a field $\mathbf{F}$, then V can be made into an $\mathbf{F}[t]$-module by setting $t \cdot v = T(v)$ for any $v \in V$. For each of the following transformations T, classify all $\mathbf{F}[t]$-submodules of V.

(a) $V = \mathbf{R}^2$ and $T(x, y) = (-y, x)$.

(b) $V = \mathbf{R}^3$ and $T(x, y, z) = (z, x, y)$.

(c) $V = \mathbf{R}^3$ and $T(x, y, z) = (y, x, 0)$.

(2) (25 Points) State the definitions of projective and injective modules. Using the definition, show that a nonzero finite abelian group is neither a projective nor an injective $\mathbf{Z}$-module.

(3) (25 Points) Let R be an integral domain. An element m of an R-module M is called a torsion element if there exists a nonzero element $r \in R$ such that $rm = 0$.

(a) If $I \subseteq R$ is a principal ideal, then prove that the R-module $I \otimes_R I$ has no torsion element other than zero.

(b) In particular, consider $R = \mathbf{Z}[x, y]$ and $I = (x, y)$. Show that the R-module $I \otimes_R I$ has a nonzero torsion element.

(4) (25 Points) Let L be the splitting field of $f(x) = x^6 + 3$ over $\mathbf{Q}$. Show that the Galois group $G = \mathrm{Gal}(L/\mathbf{Q})$ is isomorphic to S_3. Determine all proper subfields $\mathbf{Q} \subsetneq K \subsetneq L$ by using the fundamental theorem of Galois theory. For each K, find an explicit element $\alpha \in L$ so that $K = \mathbf{Q}(\alpha)$.