METU - Department of Mathematics
Graduate Preliminary Exam

Complex Analysis

Duration : 180 min. Fall 2008

Notation : $D = \{z : |z| < 1\},\ D^*(0;r) = \{z : |z| < r,\ z \neq 0\},\ \mathbb{C}^* = \mathbb{C} - \{0\}$.

1. Let $\Omega \subset \mathbb{C}$ be an open connected region and $f : \Omega \to \mathbb{C}$ be a non-constant analytic function.

   a) Show that $\det(df(z)) > 0$ for all $z \in \Omega$, except possibly for z in a discrete set $S \subset \Omega$.

   Here df is the usual differential of $f(z) = u(x,y) + iv(x,y)$ considered as a function of two variables.

   b) Show that if $z \in \Omega - S$, then there exists a neighborhood U of z in Ω such that $f|_U$ has an analytic inverse.

   b) Can you find a **bounded** region Ω and $f(z)$ analytic in Ω for which the set S is infinite ?

   Give an example, or prove that there exists no such pair (Ω, f).

2. Let $f : D^*(0;R) \to \mathbb{C}$ be a non-constant analytic function and for $a \in \mathbb{C}$ let $S_a = f^{-1}(a) \cap D^*(0;\mathbf{R/2})$.

   a) Show that there exists no such f for which the set S_a is infinite for exactly one value of a and is finite or empty for all other $a \in \mathbb{C}$.

   b) Give an example of $f : D^*(0;R) \to \mathbb{C}$ for which the set S_a is infinite for all $a \in \mathbb{C}$, except precisely for one value of a.

3. Consider the mapping $w : \mathbb{C}^* \to \mathbb{C},\ w(z) = z + 1/z$.

   a) Show that

   (i) $w(z)$ is conformal in $\mathbb{C}^*$ except at $z = \pm 1$.

   (ii) w maps the boundary of the semi-disk $U = \{z :\ |z| < 1,\ \mathrm{Im}(z) > 0\}$ onto the real axis.

   (iii) $w|_U : U \to \mathbb{C}$ is an analytic isomorphism onto $\mathcal{H}_- = \{w : \mathrm{Im}(w) < 0\}$.

b) Write the linear fractional transformation $T : \mathcal{H}_- \to D$ which satisfies the conditions $T(-3i/2) = 0,\ T(0) = i$.

(Hint : Can you determine $T^{-1}(\infty)$?).

c) Show that we obtain an analytic isomorphism $\Phi = T \circ w : U \to D$ such that $\Phi(i/2) = 0,\ \Phi'(i/2) > 0$ and that Φ is the unique isomorphism satisfying these conditions.

4. Using complex integration on the given contour Γ compute

$$\int_0^\infty \frac{dx}{x^a(1+x)}, \quad 0 < a < 1.$$

NOTE : You must specify the complex function you are integrating and justify the details of the computation.