**METU - Department of Mathematics**
**Graduate Preliminary Exam**

**Complex Analysis**

**Duration** : 180 min. Fall 2010

1. (25 pt.) Consider the following Laurent series

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{z^n} + \sum_{n=0}^{\infty} a_n z^n.$$

Suppose that $f(z)$ is analytic in the annulus $\Omega = \{z : 0 < |z| < 3\}$ and nowhere else and that $f(z)$ has a pole of order two at $z = 0$ with $Res(f; 0) = 1$.

a) Determine the coefficients $b_n,\ n \geq 1$.

b) Determine

$$lim_{n\to\infty} |a_n|^{1/n}.$$

c) Evaluate

$$\int_\gamma z^m f(z) dz$$

as $m \in \mathbb{Z}$ varies, where $\gamma$ is the square whose vertices are at $1 \pm i, -1 \pm i$ oriented counter-clockwise.

d) Can you choose the coefficients so that $f(z)$ vanishes on a sequence $\{z_n : n \geq 1\}$ which converges to $z = 0$ ? Explain.

2. (25 pt.) For any open set $\Omega \subset \mathbb{C}$ and any discrete set $S \subset \Omega$ consider the set $\mathcal{F}_\Omega(S)$ of functions analytic on $\Omega$ which vanish at each $z \in S$ to a given order $n(z)$ and nowhere else.

a) Prove that if $f, g \in \mathcal{F}_\mathbb{C}(S)$, then there exists an entire function $h(z)$ such that $f(z) = e^{h(z)} g(z)$.

b) Suppose that $S$ is a non-empty finite set and that $f \in \mathcal{F}_\mathbb{C}(S)$. Discuss the behaviour of the function $1/f(z)$ at infinity.

c) True or false ? Why ?

The statement in (a) is valid for all simply connected regions $\Omega$ and all discrete sets $S \subset \Omega$.

3. (25 pt.) a) Let $f : \Omega_{\text{op}} \subset \mathbb{C} \to \mathbb{C}$ be a meromorphic function with an isolated singularity at $z = a \in \Omega$. Prove that $z = a$ is a simple pole if and only if there exists an open neighbourhood $\Omega' \subset \Omega$ of $z = a$ and an analytic function $g : \Omega' \to \mathbb{C}$ such that $g(a) \neq 0$ and

$$f(z) = \frac{g(z)}{z-a}$$

for all $z \in \Omega'$. What is the residue of $f(z)$ at $z = a$ ?

b) Suppose that

$$\Lambda = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$$

and $\overline{T} : \Lambda \to \mathbb{C}$ is an analytic nowhere vanishing function such that

$$\overline{T}(z+1) = z\overline{T}(z)$$

for all $z \in \Lambda$. Prove that there exists a unique meromorphic function $T : \mathbb{C} \to \mathbb{C}$ with simple poles at $z = -n,\ n = 0, 1, 2, ...$ such that $T|_{\Lambda} = \overline{T}$.

c) Prove that

$$Res_{z=-n}(T(z)) = \frac{(-1)^n}{n!} T(1).$$

4. (25 pt.) a) Given $\Omega_{\text{op}} \subset \mathbb{C}$ and analytic $g : \Omega \to \mathbb{C}$, prove that for any $a \in \Omega$ the function $\Psi : \Omega \to \mathbb{C}$ defined by

$$\Psi(z) = \begin{cases} \dfrac{g(z) - g(a) - (z-a)g'(a)}{(z-a)^2} & \text{for } z \neq a \\ \frac{1}{2}g''(a) & \text{for } z = a \end{cases}$$

is analytic on $\Omega$.

b) Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and consider an analytic function $f : \Delta \to \mathbb{C}$ which satisfies $f(0) = f'(0) = 0$ and $|f(z) \leq 1$ for all $z \in \Delta$. Prove that

$$|f(z)| \leq |z|^2$$

for all $z \in \Delta$ and $|f''(0)| \leq 2$.

(Hint : Consider the function constructed from $f$ in the manner of (a) and then apply the maximum principle as in the demonstration of Schwarz's Lemma.)

c) Prove that in order for one of the inequalities in (b) to reduce to an equality even for a single $z \in \Delta$, it is necessary and sufficient that $f(z) = cz^2$ for some $c \in \mathbb{C}$ with $|c| = 1$.