Complex Analysis

PRELIMINARY EXAMINATION

Monday, 23rd September 2013
Four questions, three hours

$\boxed{1}$

[4 + 6 + 7 + 8]

(A) Let f be an entire function. If $\mathrm{Re}(f)$ is bounded, prove that f is a constant. (Consider e^f !)

(B) Let g be an entire function. If $a\,\mathrm{Re}(g) - b\,\mathrm{Im}(g)$ is bounded, where $(a, b) \neq (0, 0)$, prove that g is a constant.

(C) Show that for a not identically vanishing entire function h, the quantity $x\,\mathrm{Re}(h(z)) - y\,\mathrm{Im}(h(z))$ is unbounded, where $z = x + iy$.

(D) Does there exist an entire function w with $w'(0) = 1$ such that

$$x\,\mathrm{Re}(w(z)) + y\,\mathrm{Im}(w(z)) \leq x^2 + y^2$$

for all $z = x + iy \in \mathbb{C}$.

$\boxed{2}$

[12 + 13]

(A) Show that

$$\exp\left(\frac{z - z^{-1}}{2}\right) = \sum_{n=-\infty}^{\infty} A_n z^n$$

for every $z \in \mathbb{C} - \Big\{0\Big\}$ where,

$$A_n = \frac{1}{2\pi} \int_0^{2\pi} \cos\Big(n\theta - \sin\theta\Big) \quad .$$

(B) Show - by employing the Taylor expansion of e^w about $w = 0$ or otherwise - that

$$\int_0^{2\pi} \cos\Big(\theta + \sin\theta\Big) = \pi \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{4^k\, k!\, (k+1)!} \quad .$$

$\boxed{3}$

[5 + 7 + (4 + 9)]

(A) Let $f : \Omega \subseteq_{\text{op}} \mathbb{C} \longrightarrow \mathbb{C}$ be an analytic function. If $a \in \Omega$ is a zero of f with multiplicity $\mu \geq 2$, prove that $f'(a) = 0$.

(B) Let $g : \Omega \subseteq_{\text{op}} \mathbb{C} \longrightarrow \mathbb{C}$ be another analtytic function, $a_1, a_2, a_3, \cdots, a_N \in \Omega$ be the zeros of f with respective multiplicities $\mu_1, \mu_2, \mu_3, \cdots, \mu_N$ enclosed by the simple closed positively oriented curve Γ. Prove that

$$\frac{1}{2\pi i} \int_\Gamma \frac{g(z)\, f'(z)}{f(z)}\, dz = \sum_{n=1}^{N} g(a_n)\mu_n$$

(C) Prove that the polynomial

$$\mathrm{e}(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!}$$

has exactly n distinct rooots $b_1, b_2, b_3, \cdots, b_n \in \mathbb{C}$ and

$$\sum_{r=1}^{n} b_r^{-t} = 0$$

for every $t \in \mathbb{N}$ with $2 \leq t \leq n$.

$\boxed{4}$

[7 + 9 + 9]

(A) Let

$$\Omega = \Big\{ z \in \mathbb{C} \;\Big|\; |z| > 4 \Big\} .$$

Show that there exists an analytic function $f : \Omega \longrightarrow \mathbb{C}$ with

$$f'(z) = \frac{z}{(z-1)(z-2)(z-3)} .$$

(B) Show that there exists no analytic function $g : \Omega \longrightarrow \mathbb{C}$ with

$$g'(z) = \frac{z^2}{(z-1)(z-2)(z-3)} .$$

(C) Let

$$\Xi = \Big\{ z \in \mathbb{C} \;\Big|\; |z| > 9/4 \,,\, |z-3| > 1/8 \Big\} .$$

Does that there exists an analytic function $h : \Xi \longrightarrow \mathbb{C}$ with

$$h'(z) = \frac{z}{(z-1)(z-2)(z-3)} \qquad ?$$