M.E.T.U

Department of Mathematics

Preliminary Exam - Feb. 2011

COMPLEX ANALYSIS

Duration : 180 min.

1. (25 pt.) Consider the entire function $f(z) = e^{z^2}$.

   a) Show that for $w \in \mathbb{C}^*$ the set $f^{-1}(w) = \{z : f(z) = w\} \neq \emptyset$ and is discrete.

   b) Can you find $w \in \mathbb{C}^*$ for which the set $f^{-1}(w) = \{z : f(z) = w\}$ is bounded ? How is your answer related to the behaviour of $f(z)$ at ∞ ?

   c) Show that there exists a disc $D(0; \delta)$ such that $f(z)$ takes each value $w \in f(D(0; \delta))$ exactly twice in $D(0; \delta)$.

2. (25 pt.) Consider the set $\mathcal{F}$ of all meromorphic functions on $\mathbb{C}$ which have exactly the following zeros and poles.

   Zeros at $z_1 = 0, z_2 = 1$ of order 2, 3 respectively, poles at $p_1 = i, p_2 = -i$ each of order 2.

   a) Write a rational function $f_0(z) \in \mathcal{F}$.

   b) Determine the structure of the most general function $g(z) \in \mathcal{F}$.

   c) For an arbitrary function $g(z) \in \mathcal{F}$, $g \neq f_0$ determine the type of the singularity at each singular point in $\mathbb{C}$ of $f_0(z).g(z)$, $f_0(z) + g(z)$.

3. (25 pt.) Let $D^*(a; R)$ denote the disc of radius $R > 0$ with a puncture at the center a.

   a) Write an analytic isomorphism $\Phi : D^*(0, R_1) \to D^*(i; R_2)$.

   b) Prove that every analytic isomorphism $\Phi : D^*(0, R_1) \to D^*(i; R_2)$ extends to an analytic isomorphism $D(0, R_1) \to D(i; R_2)$.

   c) Using Φ you wrote in (a), construct an analytic isomorphism

   $$\Psi : D(0; R_1) - \{1/2\} \to D^*(i; R_2).$$

4. (25 pt.) True or false ? Explain.

a) Suppose that $f(z)$ is analytic in $D(0;\delta)$ and let $g(z) = (f(z) - 1)^N$ for some integer $N \geq 1$. If

$$\int_{\Gamma(r)} \frac{dg(z)}{g(z)} = 6\pi i N \text{ for all circles } \Gamma(r) : |z| = r, 0 < r < \delta$$

then $f(0) = 1,\ f'(0) = f''(0) = 0$ and $f'''(0) \neq 0$.

b) If $g(z)$ is analytic in $\Omega = \mathbb{C} - \{a, b\}$ and satisfies

$$\text{Residue}(g; a) = \text{Residue}(g; b) = 0$$

then for $z \in \Omega$ the integral $F(z) = \displaystyle\int_0^z g(u)du$ is independent of the path connecting 0 and z.

c) The function $f(z) = \sin(\sqrt{z^2 - 1})$ has an analytic branch in

$$\mathbb{C} - \{z \in \mathbb{R} : z \geq 1\}.$$

d) There exist non-constant doubly periodic functions $f(z)$ with simple poles at each point of the period lattice $L = \{m + ni : m, n \in \mathbb{Z}\}$.