PRELIMINARY EXAM - Feb.2012
Complex Analysis

Duration : 3 hr.

Q.1	Q.2	Q.3	Q.4	Total

1. (5 + 7 + 13 = 25 pt.) Let $n \in \mathbb{N}$ with $n \geq 2$ and $\omega = e^{\pi i/n}$.

   (A) Show that $\omega^{\frac{n(n-1)}{2}} = i^{n-1}$.

   (B) Show that $\dfrac{x^n - 1}{x - 1} = \displaystyle\prod_{k=1}^{n-1}\left(x - \omega^{2k}\right)$ for every $x \neq 1$.

   (C) Prove that

$$\prod_{k=1}^{n-1} \cos\left(\frac{k\pi}{n}\right) = \begin{cases} 0 & \text{if} \quad n \text{ is even} \\ \dfrac{(-1)^{\frac{n-1}{2}}}{2^{n-1}} & \text{if} \quad n \text{ is odd} \end{cases}$$

2. (4 + 8 + 13 = 25 pt.)

   (A) Prove that $|e^z| = e^{\mathrm{Re}(z)}$

   (B) Let f be an entire function such that $|f(z)| \leq e^{\mathrm{Re}(z)}$. Show that there exists a constant $a \in \mathbb{C}$ such that

$$f(z) = ae^z .$$

   (C) Let g be an entire function such that $g(z+1) = -g(z)$, $g(0) = 0$ and

$$|g(z)| \leq e^{\pi|\mathrm{Im}(z)|} .$$

   Show that there exists a constant $b \in \mathbb{C}$ such that

$$g(z) = b\sin\left(\pi z\right) .$$

3. (8+10+7 = 25 pt.) Let $\Omega \subset \mathbb{C}$ be a domain and $f(z)$ be a meromorphic function in Ω with a non-empty set W of poles. Choose an arbitrary point $z_0 \in \Omega - W$.

   a) Show that W is a discrete subset of Ω.

   Give an example where Ω is bounded and W is an infinite set.

   b) Show that if the residue of $f(z)$ at each pole vanishes, then

- for $z \in \Omega - W$ the integral

$$F(z) = \int_{z_0}^{z} f(u)du$$

is independent of the path $\Gamma \subset \Omega - W$ connecting z_0 and z, and

- $F(z)$ defines an analytic function in $\Omega - W$.

c) True or false ? Explain.

$F(z)$ is meromorphic in Ω with W as the set of poles.

4. (10+8+7 = 25 pt.) Let $g(z)$ be a non-constant entire periodic function, $f(z)$ be a meromorphic function in $\mathbb{C}$.

a) Let z_0 be a pole of $f(z)$. Show that the function $g \circ f$ has an essential singularity at z_0 (that is, $lim_{z \to z_0}(g \circ f(z))$ does not exist).

b) For $g(z) = e^z$, prove the result in (a) by using the argument principle.

c) Show that if $f(z)$ has at least two poles then $f \circ g$ has infinitely many poles.