Complex Analysis Preliminary Exam
Feb. 2019

1. (a) (5pts) State Rouche's Theorem.

   (b) (15 pts) Let f be a holomorphic function in $U \subset \mathbb{C}$. If $f'(z_0) = 0$ for some $z_0 \in U$ then one can find $r > 0$, $\rho > 0$ such that , $f'(z) \neq 0$, $\forall z \in \{0 < |z - z_0| < r\} \subset U$, and $|f(z) - f(z_0)| > \rho$, $\forall z : |z - z_0| = r$. (Explain why?) Then show that for any w such that $|w - f(z_0)| < \rho$, $f(z) - w$ has at least 2 **distinct** zeroes in the disk $\{|z - z_0| < r\}$.

   (c) (5 pts) Note that in part (b) you showed that if f is one-to-one in U then f' has no zeroes. Is the converse true? That is, if f' never vanishes in U, then is it true that f must be one-to-one in U ?

2. (25pts) Show that, if f is analytic on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and continuous on $\overline{\mathbb{D}}$ with $|f(z)| = 1$ on ∂D, then f is a rational function, that is $f = P/Q$ where P, Q are polynomials.

3. (a) (15pts) Find a conformal map from the strip $\{z \in \mathbb{C} : 0 < Re\ z < 1\}$ onto the unit disk $\mathbb{D}$.

   (b) (10pts) Show that $\mathbb{C}$ and the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : Imz > 0\}$ are not biholomorphic. (Two subsets are called biholomorhic if there is a conformal map between them)

4. (a) (10pts) Show that $f(z) = \sum_{n=0}^{\infty} z^{(2^n)}$ is analytic on the open unit disk.

   (b) (15pts) Show that f can not be extended analytically to any open set which is larger than the unit disk.

   (Hint: First show that $f(z) = z + f(z^2)$ and hence (why?) $f(r) \to \infty$ as $\mathbb{R} \ni r \to 1^-$. Then by considering the roots of unities, explain why this implies that f can not be extended through any point on the boundary of the unit disk.)