METU - Mathematics Department
Graduate Preliminary Exam-Fall 2007

Differentiable Manifolds

1. Let $\Phi : M \to N$ be a submanifold where $\dim(M) > 1$ and let

$$\Phi^* : C^\infty(N, \mathbb{R}) \to C^\infty(M, \mathbb{R})$$

be the restriction map $f \mapsto f \circ \Phi$.

a) Show that in general Φ^* is neither injective nor surjective.

b) Prove that if Φ is a closed imbedding then Φ^* is surjective.

2. Consider the vector field $\mathbf{v} = \dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$ on $\mathbb{R}^2$.

a) Find the integral curve of $\mathbf{v}$ through $(a, b) \in \mathbb{R}^2$.

b) Find a smooth map $\mathbb{R}^2 \to \mathbb{R}$ such that the fibers are given by the integral curves of $\mathbf{v}$.

c) Find a 1-form $\mathbf{w}$ which annihilates $\mathbf{v}$. Is $\mathbf{w}$ exact ?

3. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere with its standard smooth manifold structure. For vectors $\mathbf{a},\ \mathbf{b} \in \mathbb{R}^3$, let $\mathbf{a} \times \mathbf{b}$ and $\langle \mathbf{a}, \mathbf{b} \rangle$ respectively denote the vector product and the inner product.

a) Let $\mathbf{n}$ be the outward normal vector on S^2. Given $\sigma \in \bigwedge^1(S^2)$ defined by

$$\sigma(X) = \langle [1, 1, 1], X \times \mathbf{n} \rangle$$

prove that $\sigma = i^*(\Sigma)$ where $i : S^2 \to \mathbb{R}^3$ is the identity imbedding and

$$\Sigma = (y - z)dx + (z - x)dy + (x - y)dz.$$

b) Find $\Omega \in \bigwedge^2(\mathbb{R}^3)$ such that the volume element $\mathbf{w} \in \bigwedge^2(S^2)$ can be written in the form $\mathbf{w} = i^*(\Omega)$.

c) Does there exist $\theta \in \bigwedge^1(\mathbb{R}^3)$ such that $\mathbf{w} = i^*(d\theta)$? Explain.

4. True or false ? Explain (give a counter example if appropriate).

a) There exists no compact smooth 2-manifold M which admits an immersion $M \to \mathbb{R}^2$.

b) Let M be the compact surface and Γ be the oriented curve given in the figure. If $\mathbf{w}$ is a 1-form such that $\int_\Gamma \mathbf{w} \neq 0$, then $\mathbf{w}$ is not a closed form.

c) Let M, N be smooth manifolds with $\dim(N) > \dim(M)$ and let $\Phi : N \to M$ be a non-constant smooth map. If for some $y \in M$ the set $\Phi^{-1}(y)$ is a smooth submanifold of N, then y is a regular value of Φ.