DIFFERENTIABLE MANIFOLDS, SEPTEMBER 2010
TMS EXAM

SEPTEMBER 24, 2010

8 **1.a)** Let $\omega = (xy)\ dx \wedge dy$, a 2-form on $\mathbb{R}^2$, and $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(r,s,t) = (r - ts, r^2 s + t)$. Calculate $f^*(\omega)$, the pullback of ω by f.

8 **1.b)** Consider the vector field on the plane

$$X = 2x\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}.$$

Calculate $X(g)$ for any smooth function $g : \mathbb{R}^2 \to \mathbb{R}$.

9 **1.c)** Recall that $H^2_{DR}(S^2) = \mathbb{R}$, which is spanned by the volume form $\omega = x\ dy \wedge dz - y\ dx \wedge dz + z\ dx \wedge dy$. Using the fact that $H^1_{DR}(S^2) = 0$, show that ω cannot be written as a product of two one-forms $\omega = \alpha \wedge \beta$, which are both closed.

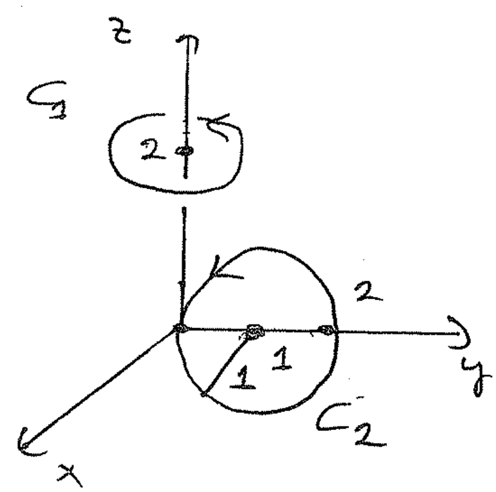
10 **2.a)** Let ω be the 1-form on $\mathbb{R}^3 - \{(x,y,z) \mid x^2 + y^2 - 1 = 0,\ z = 0\}$ given by

$$\omega = \frac{1}{2\pi}\frac{z\ d(x^2+y^2-1) - (x^2+y^2-1)\ dz}{((x^2+y^2-1)^2 + z^2)^{1/2}}.$$

Show that ω is closed.

15 **2.b)** Calculate the integral of ω over the circles shown in the figure below.

$$C_1 = \{(x,y,z) \in \mathbb{R}^3 \mid z = 2,\ x^2 + y^2 = 1\}\ ,$$
$$C_2 = \{(x,y,z) \in \mathbb{R}^3 \mid x = 0,\ (y-1)^2 + z^2 = 1\}.$$



/3 **3.a)** Prove that the subset $C = \{(x,y) \in \mathbb{R}^2 \mid y^2 = x(x-1)(x+1)\}$ is a smooth manifold by showing that $0 \in \mathbb{R}$ is a regular value for the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = y^2 - x(x-1)(x+1)$. What is its dimension? Describe its tangent space at any point $(a,b) \in C$.

/2 **3.b)** Similar to the **Part (a)** show that the unit sphere $S^2 \in \mathbb{R}^3$ is a smooth manifold of dimension two. Determine its tangent space at any point $(a,b,c) \in S^2$.

4) A one-form α on $\mathbb{R}^3$ is called a contact form if it satisfies

$$(\alpha \wedge d\alpha)(p)(e_1, e_2, e_3) > 0$$

at any point $p \in \mathbb{R}^3$, where e_i, $i = 1,2,3$, are the standard basis vectors in $T_p\mathbb{R}^3 \simeq \mathbb{R}^3$.

8 **a)** Show that the one form $\alpha = x\, dy + dz$ is a contact form on $\mathbb{R}^3$.

9 **b)** Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a map given by

$$f(x,y,z) = (a_1x + b, a_2y, a_3z),$$

where $a_1, a_2, a_3, b \in \mathbb{R}$, are some constants. Find necessary and sufficient conditions on these constants so that $f^*(\alpha) = \alpha$.

8 **c)** Show that a closed one-form ω on $\mathbb{R}^3$ cannot be a contact form.