

Differentiable Manifolds
TMS EXAM
September 16, 2013

Duration: 3 hr.

1. Find the tangent space to the surface $S : x^4 - y + z = 1$ at the point $p = (1, -1, 1)$ as a subspace of \mathbb{R}^3 in two different ways:

- (a) Using a local coordinate system at p .
- (b) Exhibiting S as the preimage of a regular value under a map $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ and then using the derivative of f (i.e. the induced map f_*).

2. Let $F : P^2(\mathbb{R}) \rightarrow P^1(\mathbb{R})$ be the map which is given by $F([x, y, z]) = [xy + x^2, y^2 + z^2]$. (Notation: The class of $x = (x_1, \dots, x_{n+1})$ in $P^n(\mathbb{R})$ is denoted by $[x] = [x_1, \dots, x_{n+1}]$.)

- (a) Show that F is well defined.
- (b) Choose a chart (U, ϕ) around a point $p = [x_0, y_0, z_0]$ in $P^2(\mathbb{R})$ with $y_0 \neq 0$ and a chart (V, ψ) around $F(p)$ with $F(U) \subset V$. Write the local expression of F in these charts. Is F smooth at p ? Why?
- (c) Compute the rank of the map F .

3. Consider the form $\omega = ydx - xdy$ in \mathbb{R}^3 .

- (a) Find the local expression of the restriction of this form to the cylinder $M : x^2 + y^2 = 1$ (i.e. $i^*(\omega)$ where $i : M \rightarrow \mathbb{R}^3$ is the inclusion map) with respect to any chart of your choice.
- (b) Let η be the form you have found in part (a). Find the local expression of $d\eta$ with respect to the chart you have used in part(a).

4. Let N be the unit ball in \mathbb{R}^3 and let f, g, h be smooth real valued functions defined on \mathbb{R}^3 . Using Stokes Theorem write the integral of $\omega = fdy \wedge dz + gdz \wedge dx + hdx \wedge dy$ (more precisely the integral of the restriction of this form) over the boundary of N as an integral over N . Also write it as a (iterated) Riemannian integral.

5. Prove the following

- (a) If $F : N \rightarrow M$ is a one-to-one immersion and N is compact, then F is an imbedding.
- (b) If $F : N \rightarrow M$ is an immersion then each $p \in N$ has a neighborhood U such that $F|U$ is an imbedding of U in M .